

Deligne Localized Functors.

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ABSTRACT. In this paper we present the notion of “Deligne localized functors”, an avatar of the derived functors, whose definition is inspired by Deligne in [SGA 4,XVII]. Their definition involves the notions of Ind and Pro categories, they always exist and are characterized in terms of universal properties. The classical localized functor, in the sense of Grothendieck and Verdier, exists if suitable conditions are verified for the Deligne localized functors. We apply these notions to triangulated and derived categories.

Introduction.

This paper arises from a study of duality theorems in algebraic geometry (Grothendieck, Hartshorne, Deligne,...) and attempts to understand the abstract structure of that theory, in order to extend it to other contexts. To that end, a preliminary study of an abstract version of the notion of derived functors turns out to be indispensable. Here we present that preliminary step, with a constant attention to its generality and possible applications to wider contexts.

We begin with an overview of the contents of this article. Paragraph §0 contains a review of notation and results on well known arguments: Ind and Pro categories, localization of categories. It is inserted only for ease of reference (which would otherwise be spread out in various papers) and to insert some specific points for which there seems to be no adequate reference. Essentially no proofs are reported, and the sources of the material are [SGA4], [SGA1], [BBD], [KS], ... The reader is advised to skip this paragraph, and to refer to it, if necessary, only in reading the principal matters at hand.

Paragraphs §1 and §2 concern the notion of derived functor, especially the Deligne definition of derived functor as sketched in [SGA4,XVII,§1]. The formalism and the ideas underlying this theory constitute our guide in the sequel. In fact we prefer the notion of localized functors, because it simplifies the terminology, without loss of any important aspect of the theory. (A similar point of view is taken independently in [KS2].) We hope that the exposition will be useful for other applications, and also for a better understanding of the theoretical status of (the usual) derived functors.

In the paragraph §3 we apply these notions to the case of triangulated categories. The principal problem here is that the categories of ind- and pro-objects of a triangulated category are no longer triangulated. Thus special care is needed to manipulate the notion of distinguished triangles.

The applications to derived categories are given in the last paragraph.

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0. Notation and Preliminaries.

0.1. Pro and Ind categories.

0.1.1. FUNCTORS CATEGORIES. For a category \mathcal{C} we put $\mathcal{C}^\vee := \mathcal{Funct}(\mathcal{C}, \mathcal{Set})$ and $\mathcal{C}^\wedge := \mathcal{Funct}(\mathcal{C}^o, \mathcal{Set})$ the categories of covariant and contravariant functors to the category of sets. We call $h^\vee : \mathcal{C}^o \rightarrow \mathcal{C}^\vee$ and $h^\wedge : \mathcal{C} \rightarrow \mathcal{C}^\wedge$ the canonical functors sending an object to its representable (covariant and contravariant) functors.

For any $F \in \mathcal{C}^\vee$ we may define the category \mathcal{C}/F (object of \mathcal{C} endowed with a morphism $h^\vee(X) \rightarrow F$, and morphisms compatible with these data); the canonical morphism $\varinjlim_{X \in \mathcal{C}/F} h^\vee(X) \rightarrow F$ (in \mathcal{C}^\vee) is an isomorphism. In particular we may describe the morphisms in \mathcal{C}^\vee between two functors as

$$\begin{aligned} \text{Hom}_{\mathcal{C}^\vee}(F, G) &= \text{Hom}_{\mathcal{C}^\vee}(\varinjlim_{X \in \mathcal{C}/F} h^\vee(X), \varinjlim_{Y \in \mathcal{C}/G} h^\vee(Y)) \\ &= \varinjlim_{X \in \mathcal{C}/F} \text{Hom}_{\mathcal{C}^\vee}(h^\vee(X), \varinjlim_{Y \in \mathcal{C}/G} h^\vee(Y)) \\ &= \varinjlim_{X \in \mathcal{C}/F} \varinjlim_{Y \in \mathcal{C}/G} h^\vee(Y)(X) \\ &= \varinjlim_{X \in \mathcal{C}/F} \varinjlim_{Y \in \mathcal{C}/G} \text{Hom}_{\mathcal{C}}(Y, X) \end{aligned}$$

which is the general formulation of the Yoneda lemma (in the last equality the standard Yoneda lemma is used).

Dually we have that $F \in \mathcal{C}^\wedge$ is isomorphic to $\varinjlim_{X \in \mathcal{C}/F} h^\wedge(X)$ and

$$\text{Hom}_{\mathcal{C}^\wedge}(F, G) = \varinjlim_{X \in \mathcal{C}/F} \varinjlim_{Y \in \mathcal{C}/G} \text{Hom}_{\mathcal{C}}(X, Y)$$

0.1.2. (PSEUDO-)FILTRANT CATEGORIES. A category \mathcal{I} is pseudo-filtrant if the following conditions hold:

- (PF1) any diagram $j \leftarrow i \rightarrow j'$ can be completed with $j \rightarrow k \leftarrow j'$ to form a commutative square;
- (PF2) any diagram $i \rightrightarrows j$ can be completed to $i \rightrightarrows j \rightarrow k$ commutative (any two parallel morphisms can be equalized).

A non-empty, pseudo-filtrant category is filtrant if it is connected (i.e. any two objects can be connected by a sequence of morphisms, independently of the directions). Note that

- (i) under the condition (PF1), a category is connected iff the following condition (C') holds: for any $i, j \in \text{ob } \mathcal{I}$ there exists $k \in \text{ob } \mathcal{I}$ and a diagram $i \rightarrow k \leftarrow j$;
- (ii) (PF2) and (C') imply (PF1);
- (iii) in particular, \mathcal{I} is filtrant iff it is non empty, (PF2) and (C') hold.

In particular, a category with amalgamed sums and cokernels is pseudo-filtrant, and a category with finite sums and cokernels is filtrant.

A filtrant category \mathcal{I} is essentially small if it admits a small full subcategory \mathcal{I}' which is cofinal, i.e. such that for any functor $F : \mathcal{I} \rightarrow \mathcal{C}$ the inclusion $i : \mathcal{I}' \rightarrow \mathcal{I}$ induces an isomorphism $\varinjlim_{\mathcal{I}'} F \circ i \leftarrow \varinjlim_{\mathcal{I}} F$ in \mathcal{C}^\wedge ; or equivalently such that for any $G : \mathcal{I}^o \rightarrow \mathcal{C}$ the inclusion $i : \mathcal{I}' \rightarrow \mathcal{I}$ induces an isomorphism $\varinjlim_{\mathcal{I}'} G \circ i \leftarrow \varinjlim_{\mathcal{I}} G$ in \mathcal{C}^\vee .

Observe that for any essentially small filtrant category \mathcal{I} there exists a small filtrant ordered set E with a cofinal functor $(E, \leq) \rightarrow \mathcal{I}$ (Deligne [RD, App. n° 1]).

0.1.3. REVERSE INDUCTIVE LIMITS. Let $F : \mathcal{I} \rightarrow \mathcal{C}$ with \mathcal{I} a small filtrant category; we define the functor “ $\varinjlim_{\mathcal{I}} F$ ” : $\mathcal{C}^o \rightarrow \mathcal{Set}$ (i.e. in \mathcal{C}^\wedge) by “ $\varinjlim_{\mathcal{I}} F = \varinjlim_{\mathcal{I}} h^\wedge(F) = \varinjlim_{i \in \mathcal{I}} h^\wedge(Fi)$ (inductive limit in the category \mathcal{C}^\wedge), i.e. (“ $\varinjlim_{\mathcal{I}} F$ ”)(X) = $\varinjlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(X, F) = \varinjlim_{\mathcal{I}} h^\vee(X)F$.”

It is representable if there exists $L \in \text{ob } \mathcal{C}$ such that $h^\wedge(L) \cong \varinjlim_{\mathcal{I}} F$, i.e. for any $W \in \text{ob } \mathcal{C}$ we have $\text{Hom}_{\mathcal{C}}(W, L) \cong \varinjlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(W, F)$, bijection realized by the universal property of (the class of) a morphism $f : L \rightarrow F(i_0)$, that is: for any $u : W \rightarrow F(i)$ there exists a unique $\varphi : W \rightarrow L$ such that the classes of $f\varphi$ and u coincide, i.e. such that there exists $i_0 \xrightarrow{s_0} k \xleftarrow{s} i$ with $F(s_0)f\varphi = F(s)u$.

The representative L is characterized by the following properties:

- (1) for any $i \in \text{ob } \mathcal{I}$ there exists $\iota_i : F(i) \rightarrow L$;
- (2) there exists $i_0 \in \text{ob } \mathcal{I}$ and a morphism $f : L \rightarrow F(i_0)$;

such that

- (a) for any i there exists $i_0 \xrightarrow{s_0} k \xleftarrow{s} i$ such that $F(s_0)f\iota_i = F(s)$;
- (b) $\iota_{i_0}f = \text{id}_L$;
- (c) for any $s : i \rightarrow j$ we have $\iota_i = \iota_j F(s)$.

In fact the bijections $\text{Hom}_{\mathcal{C}}(W, L) \rightarrow \varinjlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(W, F)$ are realized by sending φ to the class of $f\varphi$ with inverse sending the class of f_i to $\iota_i f_i$.

All functors $T : \mathcal{C} \rightarrow \mathcal{C}'$ preserve the representative L of “ \varinjlim ” F , i.e. $T(L)$ is always a representative of “ \varinjlim ” $T \circ F$.

If “ \varinjlim ” F is represented by L , then also the functor $\varinjlim_{\mathcal{I}} F$ is represented by L , using the bijection $\text{Hom}_{\mathcal{C}}(L, W) \rightarrow \varprojlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(F, W)$ sending φ to the sequence $(\varphi\iota_i)$, and inverse sending (f_i) to $f_{i_0}f$. Therefore, if the category admits inductive limits, we have necessarily $L \cong \varinjlim_{\mathcal{I}} F$, with universal data given by (1) and (c). The functor “ \varinjlim ” F is representable if and only if the canonical morphism $c : \text{“}\varinjlim\text{” } F \rightarrow \varinjlim_{\mathcal{I}} F$ (in $\text{Ind}(\mathcal{C})$, see below) is an isomorphism, i.e. if and only if there exists an inverse $f : \varinjlim_{\mathcal{I}} F \rightarrow \text{“}\varinjlim\text{” } F$ (corresponding to (2)) with $cf = \text{id}$ (corresponding to (b)) and $fc = \text{id}$ (corresponding to (a)). In that case any functor $T : \mathcal{C} \rightarrow \mathcal{C}'$ commutes with the inductive limit of the system F .

Note that $\text{Hom}_{\mathcal{C}^\wedge}(\text{“}\varinjlim\text{” } F, H) \cong \varprojlim_{\mathcal{I}} H \circ F$ and that for $F : \mathcal{I} \rightarrow \mathcal{C}$ and $G : \mathcal{J} \rightarrow \mathcal{C}$ in \mathcal{C}^\wedge we have

$$\text{Hom}_{\mathcal{C}^\wedge}(\text{“}\varinjlim\text{” } F, \text{“}\varinjlim\text{” } G) \cong \varprojlim_{\mathcal{I}} \varprojlim_{\mathcal{J}} \text{Hom}_{\mathcal{C}}(F, G) = \varprojlim_{i \in \mathcal{I}} \varprojlim_{j \in \mathcal{J}} \text{Hom}_{\mathcal{C}}(F_i, G_j).$$

0.1.4. IND-OBJECTS. We define the category of Ind-object of \mathcal{C} equivalently as either

- (i) the full subcategory $\text{Ind}\mathcal{C}$ of \mathcal{C}^\wedge whose objects are the functors isomorphic to filtrant inductive limits of representable functors; or
- (ii) the category $\text{Ind}(\mathcal{C})$ whose objects are the filtrant inductive systems, i.e. the functors $F : \mathcal{I} \rightarrow \mathcal{C}$ from a small filtrant category, and morphisms defined by $\text{Hom}_{\text{Ind}(\mathcal{C})}(F, G) = \varprojlim_{i \in \mathcal{I}} \varprojlim_{j \in \mathcal{J}} \text{Hom}_{\mathcal{C}}(F_i, G_j)$.

The equivalence $\text{Ind}(\mathcal{C}) \rightarrow \text{Ind}\mathcal{C} \subseteq \mathcal{C}^\wedge$ between these two categories is defined by sending a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ to “ \varinjlim ” F .

Then we have that the functor $h^\wedge : \mathcal{C} \rightarrow \mathcal{C}^\wedge$ extends to a left exact fully faithful functor $h^\wedge : \text{Ind}(\mathcal{C}) \rightarrow \mathcal{C}^\wedge$ and restricts to an exact fully faithful functor $h^\wedge : \mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$ making commutative the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \text{Ind}(\mathcal{C}) \\ h^\wedge \downarrow & & \downarrow h^\wedge \\ \text{Ind}\mathcal{C} & \xrightarrow{i} & \mathcal{C}^\wedge \end{array}$$

where the first morphism of any edge is an exact fully faithful functor.

We remark that in general the canonical functors $\mathcal{Funct}(\mathcal{I}, \mathcal{C}) \rightarrow \text{Ind}(\mathcal{C})$ are neither full nor faithful.

Suppose that \mathcal{C} admits filtrant inductive limits; then the canonical bijection

$$\text{Hom}_{\mathcal{C}}(\varinjlim_{\mathcal{I}} F, W) \cong \text{Hom}_{\text{Ind}(\mathcal{C})}(F, W)$$

shows that $\varinjlim_{\mathcal{I}}$ is the left adjoint of the canonical inclusion $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$. Moreover the following conditions are equivalent:

- (a) the canonical functor $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$ commutes with filtrant inductive limits;
- (b) for any $X \in \text{ob } \mathcal{C}$ the functor $h^\vee(X) \in \mathcal{C}^\vee$ commutes with filtrant inductive limits;
- (c) the functor $\varinjlim : \text{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$ is fully faithful (and so is an equivalence of categories).

0.1.5. EXTENSION OF FUNCTORS. Let $F : \mathcal{C} \rightarrow \text{Ind}\mathcal{D}$ be a functor. Then we may extend F to a functor $\overline{F} : \text{Ind}\mathcal{C} \rightarrow \text{Ind}\mathcal{D}$ uniquely defined by the condition of commutation with “ \varinjlim ”, that is $\overline{F}(\text{“}\varinjlim\text{” } X_i) := \text{“}\varinjlim\text{” } F(X_i)$. This defines a functor $\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D}) \rightarrow \mathcal{Funct}(\text{Ind}\mathcal{C}, \text{Ind}\mathcal{D})$ which is fully faithful. The image of a morphism $\varphi : F \rightarrow G$ is denoted $\overline{\varphi} : \overline{F} \rightarrow \overline{G}$.

0.1.6. DOUBLE IND CATEGORIES. The category $\text{Ind}(\mathcal{C})$ admits filtrant inductive limits, so that we have a functor “ \varinjlim ” : $\text{Ind}(\text{Ind}(\mathcal{C})) \rightarrow \text{Ind}(\mathcal{C})$ which is an exact left adjoint of the canonical inclusion. In general it is not fully faithful.

0.1.7. STRICT IND-OBJECTS. An ind-object $\varphi : \mathcal{I} \rightarrow \mathcal{C}$ is strict if \mathcal{I} is (the category associated to) a small ordered set, and one of the following equivalent condition holds:

- (i) the canonical morphisms $\varphi(i) \rightarrow \varinjlim_{\mathcal{I}} \varphi$ are monomorphisms in \mathcal{C}^\wedge ;
- (ii) for any $i \leq j$ the transition morphism $\varphi(i) \rightarrow \varphi(j)$ is a monomorphism in \mathcal{C} .

It is essentially strict if it is isomorphic in $\text{Ind}(\mathcal{C})$ to a strict one.

0.1.8. CONSTANT IND-OBJECTS. An ind-object is said to be constant if it is in the image of the canonical functor $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$, and essentially constant if it is in the essential image, i.e. if it is isomorphic in $\text{Ind}(\mathcal{C})$ to a constant one.

0.1.9. IND-REPRESENTABILITY. A functor $F \in \mathcal{C}^\wedge$ is ind-representable if it is in the essential image of the inclusion $\text{Ind}\mathcal{C} \rightarrow \mathcal{C}^\wedge$, i.e. if it is isomorphic to an inductive limit in \mathcal{C}^\wedge of representable functors.

An ind-representable functor F is left exact, i.e. the canonical morphism $F(\varinjlim_{\mathcal{I}} \varphi) \rightarrow \varinjlim_{\mathcal{I}} F\varphi$ is an isomorphism for all finite projective systems $\varphi : \mathcal{I} \rightarrow \mathcal{C}$.

0.1.10. CRITERION OF IND-REPRESENTABILITY. The following conditions are equivalent:

- (a) F is ind-representable;
- (b) the category \mathcal{C}/F is essentially small and filtrant;
- (b') if the category \mathcal{C} is equivalent to a small category: \mathcal{C}/F is filtrant;
- (c) if in \mathcal{C} all finite inductive limits are representable: F is a left exact functor and \mathcal{C}/F is essentially small;
- (c') if the category \mathcal{C} is equivalent to a small category and in \mathcal{C} all finite inductive limits are representable: F is a left exact functor.

Note that if \mathcal{C} has finite inductive limits, then F left exact implies that \mathcal{C}/F also has finite inductive limits, so that, in particular, it is filtrant.

For $F \in \mathcal{C}^\wedge$, let $\text{Sub}(F)$ be the full subcategory of \mathcal{C}/F given by the injective morphisms (i.e. the representable sub-functors of F). Then the following are equivalent:

- (i) F is strictly ind-representable (i.e. ind-representable by a strict ind-object);
- (ii) the category $\text{Sub}(F)$ is filtrant, essentially small and cofinal in \mathcal{C}/F .

0.1.11. IND-ADJOINTS. Consider $F : \mathcal{C} \rightarrow \mathcal{C}'$ a functor, and $F^\wedge : \mathcal{C}'^\wedge \rightarrow \mathcal{C}^\wedge$ the canonical inverse image; we say that F admits an ind-adjoint if one of the following equivalent conditions are satisfied:

- (a) F^\wedge sends $\text{Ind}\mathcal{C}'$ in $\text{Ind}\mathcal{C}$;
- (a') F^\wedge sends \mathcal{C}' in $\text{Ind}\mathcal{C}$;
- (b) for any $Z' \in \text{ob}\text{Ind}(\mathcal{C}')$ the functor in \mathcal{C}^\wedge sending X to $\text{Hom}_{\text{Ind}(\mathcal{C}')} (FX, Z') = \varinjlim \text{Hom}_{\mathcal{C}} (FX, Z')$ is ind-representable;
- (b') for any $X' \in \text{ob}\mathcal{C}'$ the functor in \mathcal{C}^\wedge sending X to $\text{Hom}_{\mathcal{C}} (FX, X')$ is ind-representable;
- (c) there exists a functor $G : \text{Ind}\mathcal{C}' \rightarrow \text{Ind}\mathcal{C}$ such that we have a bifunctorial isomorphism $\text{Hom}_{\text{Ind}\mathcal{C}} (X, GZ') \cong \text{Hom}_{\text{Ind}\mathcal{C}'} (FX, Z')$ for any $X \in \text{ob}\mathcal{C}$ and $Z' \in \text{ob}\text{Ind}\mathcal{C}'$;
- (c') there exists a functor $G_0 : \mathcal{C}' \rightarrow \text{Ind}\mathcal{C}$ such that we have a bifunctorial isomorphism $\text{Hom}_{\text{Ind}\mathcal{C}} (X, G_0 X') \cong \text{Hom}_{\mathcal{C}'} (FX, X')$ for any $X \in \text{ob}\mathcal{C}$ and $X' \in \text{ob}\mathcal{C}'$;
- (d) the functor $\text{Ind}(F) : \text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{C}')$ admits a right adjoint;
- (e) if \mathcal{C} is equivalent to a small category: F is right exact.

Note that if F admits a right adjoint $G' : \mathcal{C}' \rightarrow \mathcal{C}$, then it admits an ind-adjoint which is canonically isomorphic to $\text{Ind}(G')$.

0.1.12. PRESENTATION OF MORPHISMS OF IND-OBJECTS. Let $\mathcal{F}\ell(\mathcal{C})$ be the category of morphisms of the category \mathcal{C} (morphisms of $\mathcal{F}\ell(\mathcal{C})$ are the commutative squares). Then we have a canonical functor

$$\text{Ind}(\mathcal{F}\ell(\mathcal{C})) \longrightarrow \mathcal{F}\ell(\text{Ind}(\mathcal{C}))$$

which is fully faithful and admits a quasi inverse right adjoint (“parallelization of morphisms”) given by the following construction (Artin-Mazur). Let $f : X \rightarrow Y$ be a morphism with X and Y objects of $\text{Ind}(\mathcal{C})$, inductive systems indexed by $i \in I$ and $j \in J$ respectively. We say that a morphism $\varphi : X_i \rightarrow Y_j$ is a

component of f if $\varphi = f_i \in \varinjlim_j \text{Hom}_{\mathcal{C}}(X_i, Y_j)$, i.e. if the following natural diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi} & Y_j \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes in $\text{Ind}(\mathcal{C})$.

Let Φ_f be the category whose objects are the components of f , and morphisms from $\varphi : X_i \rightarrow Y_j$ to $\varphi' : X_{i'} \rightarrow Y_{j'}$ are the data of two morphisms $m : i \rightarrow i'$ (in I) and $n : j \rightarrow j'$ (in J) such that $Y_n \varphi = \varphi' X_m$ (morphisms of \mathcal{C}). Then Φ_f is a small filtering category and the natural projection functors $\Phi_f \rightarrow I$ and $\Phi_f \rightarrow J$ are cofinal functors. Therefore we may associate to f the object of $\text{Ind}(\mathcal{F}(\mathcal{C}))$ given by the system $\varphi : X_\varphi = X_i \rightarrow Y_j = Y_\varphi$ indexed by φ in the category Φ_f . We may prove the adjoint property and the functoriality of the construction. More precisely, the construction gives the Ind-adjoint of the canonical functor $\mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{F}(\text{Ind}(\mathcal{C}))$.

In the same way (see [AM]) we can prove the UNIFORM APPROXIMATION LEMMA: let Δ be a finite type of diagram with commutativity conditions without loops (i.e. a finite category without loops), and denote by $\Delta(\mathcal{C})$ the category of diagrams of type Δ in \mathcal{C} , that is, the category of functors from Δ to \mathcal{C} ; then the natural functor

$$\text{Ind}(\Delta(\mathcal{C})) \longrightarrow \Delta(\text{Ind}(\mathcal{C}))$$

admits a right adjoint which is a quasi-inverse.

0.1.13. The previous construction can be dualized in order to define the category of pro-objects as $\text{Pro}(\mathcal{C}) := (\text{Ind}(\mathcal{C}^o))^o$. In the following we make explicit all the definitions and results.

0.1.14. REVERSE PROJECTIVE LIMITS. Let $F : \mathcal{I}^o \rightarrow \mathcal{C}$ with \mathcal{I} a small filtrant category; we define the functor “ \varprojlim ” $_{\mathcal{I}} F : \mathcal{C} \rightarrow \mathcal{S}et$ (i.e. in \mathcal{C}^\vee) by “ \varprojlim ” $_{\mathcal{I}} F = \varprojlim_{\mathcal{I}} h^\vee(F) = \varprojlim_{i \in \mathcal{I}} h^\vee(Fi)$ (inductive limit in the category \mathcal{C}^\vee), i.e. (“ \varprojlim ” $_{\mathcal{I}} F$)(X) = $\varprojlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(F, X) = \varprojlim_{\mathcal{I}} h^\vee(X)F$.

This “ \varprojlim ” is representable if there exists $M \in \text{ob } \mathcal{C}$ such that $h^\vee(M) \cong \text{“}\varprojlim\text{”}_{\mathcal{I}} F$, i.e. for any $W \in \text{ob } \mathcal{C}$ we have $\text{Hom}_{\mathcal{C}}(M, W) \cong \varprojlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(F, W)$, with the bijection being realized by the universal property of (the class of) a morphism $f : F(i_0) \rightarrow M$: for any $u : F(i) \rightarrow W$ there exists a unique $\varphi : M \rightarrow W$ such that the classes of φf and u coincide, i.e. such that there exists $i_0 \xrightarrow{s_0} k \xleftarrow{s} i$ with $\varphi f F(s_0) = u F(s)$.

The representative M is characterized by the following properties:

- (1) for any $i \in \text{ob } \mathcal{I}$ there exists $\pi_i : M \rightarrow F(i)$;
- (2) there exists $i_0 \in \text{ob } \mathcal{I}$ and a morphism $f : F(i_0) \rightarrow M$;

such that

- (a) for any i there exists $i_0 \xrightarrow{s_0} k \xleftarrow{s} i$ such that $\pi_i f F(s_0) = F(s)$;
- (b) $f \pi_{i_0} = \text{id}_M$;
- (c) for any $s : i \rightarrow j$ we have $\pi_i = F(s) \pi_j$.

In fact the bijections $\text{Hom}_{\mathcal{C}}(M, W) \rightarrow \varprojlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(F, W)$ are realized by sending φ to the class of φf with inverse sending the class of f_i to $f_i \pi_i$.

All functors $T : \mathcal{C} \rightarrow \mathcal{C}'$ preserve the representative M of “ \varprojlim ” $_{\mathcal{I}} F$, i.e. $T(L)$ is always a representative of “ \varprojlim ” $_{\mathcal{I}} T \circ F$.

If “ \varprojlim ” $_{\mathcal{I}} F$ is represented by M , then also the functor “ \varprojlim ” $_{\mathcal{I}} F$ is represented by M , using the bijection $\text{Hom}_{\mathcal{C}}(W, M) \rightarrow \varprojlim_{\mathcal{I}} \text{Hom}_{\mathcal{C}}(W, F)$ which sends φ to the sequence $(\pi_i \varphi)$, with inverse sending (f_i) to $f f_{i_0}$. Therefore, if the category admits projective limits, we have necessarily $L \cong \varprojlim_{\mathcal{I}} F$, with universal data given by (1) and (c). The functor “ \varprojlim ” $_{\mathcal{I}} F$ is representable if and only if the canonical morphism $c : \varprojlim_{\mathcal{I}} F \rightarrow \text{“}\varprojlim\text{”}_{\mathcal{I}} F$ (in $\text{Pro}(\mathcal{C})$, see below) is an isomorphism, i.e. if and only if there exists an inverse $f : \text{“}\varprojlim\text{”}_{\mathcal{I}} F \rightarrow \varprojlim_{\mathcal{I}} F$ (corresponding to (2)) with $f c = \text{id}$ (corresponding to (b)) and $c f = \text{id}$ (corresponding to (a)).

Note that $\text{Hom}_{\mathcal{C}^\vee}(\text{“}\varprojlim\text{”}_{\mathcal{I}} F, H) \cong \varprojlim_{\mathcal{I}} H \circ F$ and that for $F : \mathcal{I}^o \rightarrow \mathcal{C}$ and $G : \mathcal{J}^o \rightarrow \mathcal{C}$ we have

$$\text{Hom}_{\mathcal{C}^\vee}(\text{“}\varprojlim\text{”}_{\mathcal{I}} F, \text{“}\varprojlim\text{”}_{\mathcal{J}} G) \cong \varprojlim_{\mathcal{I}} \varprojlim_{\mathcal{J}} \text{Hom}_{\mathcal{C}}(G, F) .$$

0.1.15. PRO-OBJECTS. We define the category of Pro-object of \mathcal{C} (anti)equivalently as:

- (i) the full subcategory $\text{Pro}\mathcal{C}$ of \mathcal{C}^\vee whose objects are the functors isomorphic to filtrant inductive limits of representable functors;
- (ii) the category $\text{Pro}(\mathcal{C})$ whose objects are the filtrant projective systems, i.e. the functors $F : \mathcal{I}^o \rightarrow \mathcal{C}$ from a small filtrant category, and morphisms defined by $\text{Hom}_{\text{Pro}(\mathcal{C})}(F, G) = \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(Fi, Gj)$.

The (anti)equivalence $\text{Pro}(\mathcal{C})^o \rightarrow \text{Pro}\mathcal{C} \subseteq \mathcal{C}^\vee$ between the two categories is defined by sending a functor $F : \mathcal{I}^o \rightarrow \mathcal{C}$ to “ $\varprojlim_{\mathcal{I}} F$ ”.

Then we have that the functor $h^\vee : \mathcal{C}^o \rightarrow \mathcal{C}^\vee$ extends to a left exact fully faithful functor $h^\vee : \text{Pro}(\mathcal{C})^o \rightarrow \mathcal{C}^\vee$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{C}^o & \xrightarrow{i^o} & \text{Pro}(\mathcal{C})^o \\ h^\vee \downarrow & & \downarrow h^\vee \\ \text{Pro}\mathcal{C} & \xrightarrow{i} & \mathcal{C}^\vee \end{array}$$

where the first morphism of any edge is an exact fully faithful functor.

Observe that in general the canonical functors $\mathcal{Funct}(\mathcal{I}^o, \mathcal{C}) \rightarrow \text{Pro}(\mathcal{C})$ are neither full nor faithful.

Suppose that \mathcal{C} admits filtrant projective limits; then the canonical bijection $\text{Hom}_{\mathcal{C}}(W, \varprojlim_{\mathcal{I}} F) \cong \text{Hom}_{\text{Pro}(\mathcal{C})}(W, F)$ shows that $\varprojlim_{\mathcal{I}} F$ is the right adjoint of the canonical inclusion $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$. Moreover the following conditions are equivalent:

- (a) the canonical functor $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$ commutes with filtrant inductive limits;
- (b) for any $X \in \text{ob}\mathcal{C}$ the functor $h^\vee(X) \in \mathcal{C}^\vee$ commutes with filtrant inductive limits;
- (c) the functor $\varprojlim : \text{Pro}(\mathcal{C}) \rightarrow \mathcal{C}$ is fully faithful (and so an equivalence of categories).

0.1.16. EXTENSION OF FUNCTORS. Let $F : \mathcal{C} \rightarrow \text{Pro}\mathcal{D}$ be a functor. Then we may extend F to a functor $\overline{F} : \text{Pro}\mathcal{C} \rightarrow \text{Pro}\mathcal{D}$ uniquely defined by the condition of commutation with “ \varprojlim ”, that is $\overline{F}(\varprojlim_{\mathcal{I}} X_i) := \varprojlim_{\mathcal{I}} \overline{F}(X_i)$. This defines a functor $\mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{D}) \rightarrow \mathcal{Funct}(\text{Pro}\mathcal{C}, \text{Pro}\mathcal{D})$ which is fully faithful. The image of a morphism $\varphi : F \rightarrow G$ is denoted $\overline{\varphi} : \overline{F} \rightarrow \overline{G}$.

0.1.17. DOUBLE PRO CATEGORIES. The category $\text{Pro}(\mathcal{C})$ admits filtrant inductive limits, so that we have a functor “ \varprojlim ” : $\text{Pro}(\text{Pro}(\mathcal{C})) \rightarrow \text{Pro}(\mathcal{C})$ which is an exact right adjoint of the canonical inclusion. In general it is not fully faithful.

0.1.18. STRICT PRO-OBJECTS. A pro-object $\varphi : \mathcal{I}^o \rightarrow \mathcal{C}$ is strict if \mathcal{I} is (the category associated to) a small ordered set, and one of the following equivalent condition holds:

- (i) the canonical morphisms “ $\varprojlim_{\mathcal{I}} \varphi \rightarrow \varphi(i)$ ” are epimorphisms in \mathcal{C}^\vee ;
- (ii) for any $i \leq j$ the transition morphism $\varphi(j) \rightarrow \varphi(i)$ is an epimorphism in \mathcal{C} .

The pro-object φ is essentially strict if it is isomorphic in $\text{Pro}(\mathcal{C})$ to a strict pro-object.

0.1.19. CONSTANT PRO-OBJECTS. A pro-object is said to be constant if it is in the image of the canonical functor $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$, and essentially constant if it is in the essential image, i.e., if it is isomorphic in $\text{Pro}(\mathcal{C})$ to a constant pro-object.

0.1.20. PRO-REPRESENTABILITY. A functor $F \in \mathcal{C}^\vee$ is pro-representable if it is in the essential image of the inclusion $\text{Pro}\mathcal{C} \rightarrow \mathcal{C}^\vee$, i.e., if it is isomorphic to an inductive limit in \mathcal{C}^\vee of representable functors.

A pro-representable functor F is left exact, i.e. the canonical morphism $F(\varprojlim_{\mathcal{I}} \varphi) \rightarrow \varprojlim_{\mathcal{I}} F\varphi$ is an isomorphism for all finite projective system $\varphi : \mathcal{I} \rightarrow \mathcal{C}$.

0.1.21. CRITERION OF PRO-REPRESENTABILITY. The following conditions are equivalent:

- (a) F is pro-representable;
- (b) the category \mathcal{C}/F is essentially small and filtrant;
- (b') if the category \mathcal{C} is equivalent to a small category: \mathcal{C}/F is filtrant;
- (c) if in \mathcal{C} the finite projective limits are representable: F is a left exact functor and \mathcal{C}/F is essentially small;
- (c') if the category \mathcal{C} is equivalent to a small category and in \mathcal{C} the finite projective limits are representable: F is a left exact functor.

We remark that if \mathcal{C} has finite projective limits, then F left exact implies that \mathcal{C}/F also has finite inductive limits, so that, in particular, it is filtrant.

For $F \in \mathcal{C}^\vee$, let $\text{Sub}(F)$ be the full subcategory of \mathcal{C}/F given by the injective morphisms (i.e. the representable sub-functors of F). Then the following are equivalent:

- (i) F is strictly pro-representable (i.e. pro-representable by a strict pro-object);
- (ii) the category $\text{Sub}(F)$ is filtrant, essentially small and cofinal in \mathcal{C}/F .

0.1.22. PRO-ADJOINTS. Consider $F : \mathcal{C} \rightarrow \mathcal{C}'$ a functor, and $F^\vee : \mathcal{C}'^\vee \rightarrow \mathcal{C}^\vee$ the canonical inverse image; we say that F admits a pro-adjoint if one of the following equivalent conditions are satisfied:

- (a) F^\vee sends $\text{Pro}\mathcal{C}'$ in $\text{Pro}\mathcal{C}$;
- (a') F^\vee sends \mathcal{C}' in $\text{Pro}\mathcal{C}$;
- (b) for any $Z' \in \text{ob}\text{Pro}(\mathcal{C}')$ the functor in \mathcal{C}^\vee sending X to $\text{Hom}_{\text{Ind}(\mathcal{C}')} (Z', FX) = \varinjlim \text{Hom}_{\mathcal{C}} (Z', FX)$ is pro-representable;
- (b') for any $X' \in \text{ob}\mathcal{C}'$ the functor in \mathcal{C}^\vee sending X to $\text{Hom}_{\mathcal{C}} (X', FX)$ is pro-representable;
- (c) there exists a functor $G : \text{Pro}\mathcal{C}' \rightarrow \text{Pro}\mathcal{C}$ such that we have a bifunctorial isomorphism $\text{Hom}_{\text{Pro}\mathcal{C}} (GZ', X) \cong \text{Hom}_{\text{Pro}\mathcal{C}'} (Z', FX)$ for any $X \in \text{ob}\mathcal{C}$ and $Z' \in \text{ob}\text{Ind}\mathcal{C}'$;
- (c') there exists a functor $G_0 : \mathcal{C}' \rightarrow \text{Pro}\mathcal{C}$ such that we have a bifunctorial isomorphism $\text{Hom}_{\text{Pro}\mathcal{C}} (G_0X', X) \cong \text{Hom}_{\mathcal{C}'} (X', FX)$ for any $X \in \text{ob}\mathcal{C}$ and $X' \in \text{ob}\mathcal{C}'$;
- (d) the functor $\text{Pro}(F) : \text{Pro}(\mathcal{C}) \rightarrow \text{Pro}(\mathcal{C}')$ admits a left adjoint;
- (e) if \mathcal{C} is equivalent to a small category: F is left exact.

Remark that if F admits a left adjoint $G' : \mathcal{C}' \rightarrow \mathcal{C}$, then it admits a pro-adjoint which is canonically isomorphic to $\text{Pro}(G')$.

0.1.23. PRESENTATION OF MORPHISMS OF PRO-OBJECTS. Let again $\mathcal{F}\ell(\mathcal{C})$ be the category of morphisms of the category \mathcal{C} (morphisms of $\mathcal{F}\ell(\mathcal{C})$ are the commutative squares). Then we have a canonical functor

$$\text{Pro}(\mathcal{F}\ell(\mathcal{C})) \longrightarrow \mathcal{F}\ell(\text{Pro}(\mathcal{C}))$$

which is fully faithful and admits a quasi inverse left adjoint (“parallelization of morphisms”) given by the dual construction to that of 0.1.12.

Also the UNIFORM APPROXIMATION LEMMA holds: let Δ be a finite type of diagram with commutativity conditions without loops (i.e. a finite category without loops), and let $\Delta(\mathcal{C})$ denote the category of diagrams of type Δ in \mathcal{C} , that is the category of functors from Δ to \mathcal{C} ; then the natural functor

$$\text{Pro}(\Delta(\mathcal{C})) \longrightarrow \Delta(\text{Pro}(\mathcal{C}))$$

admits a left adjoint which is a quasi-inverse.

0.1.24. Occasionally we will need categories such as $\text{Pro}\text{Ind}\mathcal{C}$ or $\text{Ind}\text{Pro}\mathcal{C}$. We remark only that the $\text{Hom}_{\mathcal{C}}$ as a bifunctor on $\mathcal{C}^o \times \mathcal{C}$ can be extended to a bifunctor

$$\text{Ind}\text{Hom}_{\mathcal{C}} : (\text{Pro}\mathcal{C})^o \times \text{Ind}\mathcal{C} \longrightarrow \text{IndSet}$$

as $\text{Ind}\text{Hom}_{\mathcal{C}}((X_i), (Y_j)) = \varinjlim_{i \in I} \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(X_i, Y_j)$ and its composition with the inductive limit functor of Set , i.e. $\varinjlim_{i \in I} \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(X_i, Y_j)$ is just the restriction of $\text{Hom}_{\text{Pro}\text{Ind}\mathcal{C}}$ to the (full) subcategories $\text{Pro}\mathcal{C}$ (first argument) and $\text{Ind}(\mathcal{C})$ (second argument).

Similar remarks hold for the bifunctor $\text{Pro}\text{Hom}_{\mathcal{C}}$ with respect to the category $\text{Ind}\text{Pro}\mathcal{C}$.

In particular, we remark that for $(X_i)_{i \in I}$ in $\text{Pro}\mathcal{C}$ and $(Y_j)_{j \in J}$ in $\text{Ind}\mathcal{C}$ we have the equalities

$$\text{Hom}_{\text{Ind}\text{Pro}\mathcal{C}}((X_i), (Y_j)) = \varinjlim_{i \in I} \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(X_i, Y_j) = \varinjlim_{j \in J} \varinjlim_{i \in I} \text{Hom}_{\mathcal{C}}(X_i, Y_j) = \text{Hom}_{\text{Pro}\text{Ind}\mathcal{C}}((X_i), (Y_j))$$

and

$$\text{Hom}_{\text{Ind}\text{Pro}\mathcal{C}}((Y_j), (X_i)) = \varprojlim_{j \in J} \varprojlim_{i \in I} \text{Hom}_{\mathcal{C}}(Y_j, X_i) = \varprojlim_{i \in I} \varprojlim_{j \in J} \text{Hom}_{\mathcal{C}}(Y_j, X_i) = \text{Hom}_{\text{Pro}\text{Ind}\mathcal{C}}((Y_j), (X_i)) .$$

0.1.25. GENERALIZED ADJUNCTIONS. We say that $F : \mathcal{C} \rightarrow \text{Pro}\mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \text{Ind}\mathcal{C}$ are generalized adjoints if there is a bifunctorial isomorphism

$$\text{Hom}_{\text{Pro}\mathcal{C}'}(FX, X') \cong \text{Hom}_{\text{Ind}\mathcal{C}}(X, GX')$$

for any $X \in \text{ob}\mathcal{C}$ and $X' \in \text{ob}\mathcal{C}'$. In that case for any $X \in \text{ob}\text{Pro}\mathcal{C}$ and $X' \in \text{ob}\text{Ind}\mathcal{C}'$ we have bifunctorial isomorphisms

$$\text{Ind}\text{Hom}_{\mathcal{C}'}(FX, X') \cong \text{Ind}\text{Hom}_{\mathcal{C}}(X, GX')$$

and

$$\mathrm{Hom}_{\mathrm{Pro} \, \mathrm{Ind} \, \mathcal{C}'}(FX, X') \cong \mathrm{Hom}_{\mathrm{Ind} \, \mathrm{Pro} \, \mathcal{C}}(X, GX') .$$

Moreover each of the two functors F and G determines the other, up to isomorphisms.

We also have the dual notions of generalized coadjoint functors: $F : \mathcal{C} \rightarrow \mathrm{Ind} \, \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathrm{Pro} \, \mathcal{C}$ are generalized coadjoint if there is a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathrm{Ind} \, \mathcal{C}'}(FX, X') \cong \mathrm{Hom}_{\mathrm{Pro} \, \mathcal{C}}(X, GX')$$

for any $X \in \mathrm{ob} \, \mathcal{C}$ and $X' \in \mathrm{ob} \, \mathcal{C}'$. In that case for any $X \in \mathrm{ob} \, \mathrm{Ind} \, \mathcal{C}$ and $X' \in \mathrm{ob} \, \mathrm{Pro} \, \mathcal{C}'$ we have bifunctorial isomorphisms

$$\mathrm{Pro} \, \mathrm{Hom}_{\mathcal{C}'}(FX, X') \cong \mathrm{Pro} \, \mathrm{Hom}_{\mathcal{C}}(X, GX')$$

and

$$\mathrm{Hom}_{\mathrm{Ind} \, \mathrm{Pro} \, \mathcal{C}'}(FX, X') \cong \mathrm{Hom}_{\mathrm{Pro} \, \mathrm{Ind} \, \mathcal{C}}(X, GX') .$$

Moreover each of the two functors F and G determines the other, up to isomorphisms. Note however that the notion of coadjointness is not useful when the categories admit (filtrant) inductive and projective limits, since it then reduces to the usual notion of adjunction.

0.1.26. INTERSECTION OF Ind AND Pro. The canonical square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathrm{Pro}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Ind}(\mathcal{C}) & \longrightarrow & \mathrm{Pro}(\mathrm{Ind}(\mathcal{C})) \end{array}$$

is cartesian in the following sense: an object of $\mathrm{Pro}(\mathrm{Ind}(\mathcal{C}))$ which is in the image either of $\mathrm{Pro}(\mathcal{C})$ or of $\mathrm{Ind}(\mathcal{C})$ is really in \mathcal{C} ; i.e. $\mathrm{Pro}(\mathcal{C}) \cap \mathrm{Ind}(\mathcal{C})$ in $\mathrm{Pro}(\mathrm{Ind}(\mathcal{C}))$ is just \mathcal{C} .

0.2. Multiplicative systems and localization.

0.2.1. RIGHT AND LEFT MULTIPLICATIVE SYSTEMS. Let \mathcal{C} be a category; a multiplicative system in \mathcal{C} is a family S of morphism of \mathcal{C} such that:

(S1) $\mathrm{id}_X \in S$ for any X in \mathcal{C} ;

(S2) if $f, g \in S$ then $g \circ f \in S$ if it exists.

A multiplicative system is said to be right (resp. left) if the following conditions are satisfied:

(S3) we may complete any diagram with $s \in S$

$$\begin{array}{ccc} & Z & W \xrightarrow{g} Z \\ & \downarrow s & \downarrow t \\ X \xrightarrow{f} Y & \text{in} & X \xrightarrow{f} Y \end{array}$$

commutative with $t \in S$; (resp. dually with the arrows reversed);

(S4) for any two morphisms $f, g : X \rightarrow Y$ in \mathcal{C} consider the following conditions:

(i) there exists $s \in S$, $s : W \rightarrow X$ such that $f \circ s = g \circ s$;

(ii) there exists $t \in S$, $t : Y \rightarrow Z$ such that $t \circ f = t \circ g$;

then (i) implies (ii) (resp. (ii) implies (i))

A right and left multiplicative system is said to be bilateral, or simply a multiplicative system, if there is no possibility of confusion.

0.2.2. QUASI-SATURATED MULTIPLICATIVE SYSTEMS. A multiplicative system S is right (resp. left) quasi-saturated if the following condition holds: if $g \circ f \in S$ and $f \in S$ then $g \in S$ (resp. if $g \circ f \in S$ and $g \in S$ then $f \in S$). A right and left quasi-saturated multiplicative system is said to be quasi-saturated.

0.2.3. LOCALIZED CATEGORIES. Let S be a multiplicative system in \mathcal{C} ; we define the localized category of \mathcal{C} with respect to S , denoted by \mathcal{C}_S or $\mathcal{C}[S^{-1}]$, to be a category endowed with a functor $Q : \mathcal{C} \rightarrow \mathcal{C}_S$ such that for any $s \in S$ the image $Q(s)$ is an isomorphism, and which is universal for this property: for any category \mathcal{D} with a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(s)$ is isomorphism for any $s \in S$, then there exists a unique factorization:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{C}_S \\ F \searrow & & \downarrow \exists! \\ & & \mathcal{D} . \end{array}$$

More generally we may solve the same universal problem for any family of morphisms of a category, but the localized category has a rather complicated construction.

If S is a right (resp. left) multiplicative system in \mathcal{C} , then the category \mathcal{C}_S is constructed in the following way: the objects of \mathcal{C}_S are the objects of \mathcal{C} ; the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \varinjlim_{(Y', t') \in Y/S} \mathrm{Hom}_{\mathcal{C}}(X, Y')$$

where Y/S is the full subcategory of Y/\mathcal{C} given by the objects $t' : Y \rightarrow Y'$ with $t' \in S$ (resp.

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \varinjlim_{(X', s') \in S/X} \mathrm{Hom}_{\mathcal{C}}(X', Y)$$

where S/X is the full subcategory of \mathcal{C}/X given by the objects $s' : X' \rightarrow X$, with $s' \in S$). The functor Q is defined in the obvious way.

If the multiplicative system is bilateral, the more symmetric formula

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \varinjlim_{\substack{(X', s') \in S/X \\ (Y', t') \in Y/S}} \mathrm{Hom}_{\mathcal{C}}(X', Y')$$

also works.

0.2.4. LOCALIZATION OF TRIANGULATED CATEGORIES. Let (\mathcal{T}, T) be a triangulated category; a null system \mathcal{N} of \mathcal{T} is a family of objects of \mathcal{T} such that:

- (N1) $0 \in \mathcal{N}$,
- (N2) $N \in \mathcal{N}$ if and only if $TN \in \mathcal{N}$,
- (N3) if $X \rightarrow Y \rightarrow Z \rightarrow TX$ is a distinguished triangle and $X, Y \in \mathcal{N}$, then $Z \in \mathcal{N}$.

Notice that the shift property of \mathcal{T} and (N2) permit us to extend (N3): if two vertices of a distinguished triangle are in \mathcal{N} , then so is the third vertex.

In the triangulated category we can localize with respect to a null system; in fact the family of morphisms

$$S(\mathcal{N}) = \left\{ f : X \rightarrow Y \mid \exists \text{ dist.tr. } X \xrightarrow{f} Y \rightarrow N \rightarrow TX \text{ with } N \in \mathcal{N} \right\}$$

is a quasi-saturated multiplicative system in \mathcal{T} . Moreover $S(\mathcal{N})$ satisfies the following two properties:

- (ST1) $s \in S(\mathcal{N})$ if and only if $T(s) \in S(\mathcal{N})$;
- (ST2) if two arrows of a morphism of distinguished triangles are in $S(\mathcal{N})$, then so too is the third arrow.

UNIVERSAL PROPERTY: put $\mathcal{T}/\mathcal{N} = \mathcal{T}_{S(\mathcal{N})}$ and let $Q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{N}$ be the canonical functor; then \mathcal{T}/\mathcal{N} is canonically a triangulated category; $Q(N) \cong 0$ for any $N \in \mathcal{N}$ and \mathcal{T}/\mathcal{N} is universal with respect to this property in the category of triangulated categories.

EXAMPLE: If \mathcal{A} is an abelian category and $H : \mathcal{T} \rightarrow \mathcal{A}$ is a cohomological functor, then the class $\mathcal{N}_H = \{X \in \mathcal{T} \mid H(T^n X) = 0 \ \forall n \in \mathbb{Z}\}$ of H -acyclic objects is a null system. Remark that $S(\mathcal{N}_H) = \{f \mid H(T^n f) \text{ iso } \forall n \in \mathbb{Z}\}$.

1. Deligne localized functors.

1.1. DEFINITION (LOCALIZING FUNCTORS). Let \mathcal{C} be a category and S a quasi-saturated right multiplicative system in \mathcal{C} . Then we define the right localizing functor with respect to S as $r'_S : \mathcal{C} \rightarrow \mathrm{Ind} \mathcal{C}$ by

$$r'_S(X) := \varinjlim_{s : X \rightarrow X'} X'$$

where the index category is X/S (morphisms in S with source X), as an inductive system, that is

$$r'_S(X)(Z) := \varinjlim_{s : X \rightarrow X'} \mathrm{Hom}_{\mathcal{C}}(Z, X')$$

as functor $\mathcal{C}^o \rightarrow \mathrm{Set}$.

Dually, if S a quasi-saturated left multiplicative system in \mathcal{C} , we define the left localizing functor with respect to S as $l'_S : \mathcal{C} \rightarrow \mathrm{Pro} \mathcal{C}$ by

$$l'_S(X) := \varprojlim_{s : X' \rightarrow X} X'$$

where the index category is S/X (morphisms in S with target X), as a projective system, that is

$$l'_S(X)(Z) := \varprojlim_{s: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Z)$$

as functor $\mathcal{C} \rightarrow \text{Set}$.

1.1.1. ACTION ON MORPHISMS. If $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then the morphism $r'_S(f) : r'_S(X) \rightarrow r'_S(Y)$ is defined in

$$\text{Hom}_{\text{Ind}\mathcal{C}}(r'_S(X), r'_S(Y)) = \varprojlim_{s: X \rightarrow X'} \varinjlim_{t: Y' \rightarrow Y} \text{Hom}_{\mathcal{C}}(X', Y')$$

by the following construction: for any $s : X \rightarrow X'$ we complete the diagram with $f : X \rightarrow Y$ to a square

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{t} & Y' \end{array}$$

with $t \in S$; then (t, f') is a representative in $\varinjlim_{t: Y' \rightarrow Y} \text{Hom}_{\mathcal{C}}(X', Y')$ of the component of $r'_S(f)$ in s . That the definition is well-posed, that is independent of the choices made in completing the square, follows from the properties of right saturated multiplicative systems.

From the point of view of functors, the construction gives a morphism $r'_S(f) : r'_S(X) \rightarrow r'_S(Y)$ whose evaluation in Z is

$$r'_S(f)(Z) : r'_S(X)(Z) = \varprojlim_{s: X \rightarrow X'} \text{Hom}_{\mathcal{C}}(Z, X') \longrightarrow \varinjlim_{t: Y' \rightarrow Y} \text{Hom}_{\mathcal{C}}(Z, Y') = r'_S(Y)(Z)$$

which sends (s, φ) with $\varphi : Z \rightarrow X'$ to $(t, f'\varphi)$, a well-defined element in the inductive limit on the right hand side.

Dually, if $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then the morphism $l'_S(f) : l'_S(X) \rightarrow l'_S(Y)$ is defined in

$$\text{Hom}_{\text{Pro}\mathcal{C}}(l'_S(X), l'_S(Y)) = \varprojlim_{t: Y' \rightarrow Y} \varinjlim_{s: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Y')$$

by the following construction: for any $t : Y' \rightarrow Y$ we complete the diagram with $f : X \rightarrow Y$ to a square

$$\begin{array}{ccc} X & \xleftarrow{s} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xleftarrow{t} & Y' \end{array}$$

with $s \in S$; then (s, f') is a representative in $\varinjlim_{s: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Y')$ of the component of $l'_S(f)$ in t .

From the point of view of functors, the construction gives a morphism $l'_S(f) : l'_S(X) \rightarrow l'_S(Y)$ whose evaluation in Z is

$$l'_S(f)(Z) : l'_S(Y)(Z) = \varinjlim_{t: Y' \rightarrow Y} \text{Hom}_{\mathcal{C}}(Y', Z) \longrightarrow \varprojlim_{s: X' \rightarrow X} \text{Hom}_{\mathcal{C}}(X', Z) = l'_S(X)(Z)$$

which sends (t, ψ) with $\psi : Y' \rightarrow Z$ to $(s, \psi f')$, a well-defined element in the inductive limit.

1.1.2. LEMMA. Suppose $f : X \rightarrow Y$ in \mathcal{C} is a morphism in S , then $r'_S(f) : r'_S(X) \rightarrow r'_S(Y)$ is an isomorphism in $\text{Ind}\mathcal{C}$. Dually, $l'_S(f) : l'_S(X) \rightarrow l'_S(Y)$ is an isomorphism in $\text{Pro}\mathcal{C}$.

PROOF. In fact we can define the inverse morphism $s(f) : r'_S(Y) \rightarrow r'_S(X)$ using the following construction: for any $t : Y \rightarrow Y'$ in S , the composition with f gives $tf : X \rightarrow Y'$ in S , and this is the component of $s(f)$ in t . In terms of functor morphisms, this is the morphism $s(f)(Z) : r'_S(Y)(Z) \rightarrow r'_S(X)(Z)$ sending (t, ψ) to (tf, ψ) . It is well defined, and the compositions with $r_S(f)$ are clearly the identities (one composition is easy, the other requires the quasi-saturatedness of the multiplicative system). \square

1.1.3. PROPOSITION. In particular we can extend $r'_S : \mathcal{C} \rightarrow \text{Ind}\mathcal{C}$ to a functor $r_S : \mathcal{C}_S \rightarrow \text{Ind}\mathcal{C}$ and we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r'_S} & \text{Ind}\mathcal{C} \\ Q \downarrow & \nearrow r_S & \\ \mathcal{C}_S & & \end{array}$$

Moreover the functor r_S is the ind-adjoint of the canonical functor $Q : \mathcal{C}_S \rightarrow \mathcal{C}$.

Dually, we can extend l'_S to a functor $l_S : \mathcal{C}_S \rightarrow \text{Pro}\mathcal{C}$ and we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{l'_S} & \text{Pro}\mathcal{C} \\ Q \downarrow & \nearrow l_S & \\ \mathcal{C}_S & & \end{array}$$

Moreover the functor l_S is the pro-adjoint of the canonical functor $Q : \mathcal{C}_S \rightarrow \mathcal{C}$.

PROOF. The first claim is an obvious consequences of the lemma and the universal property of the localized category. The last claim is a consequence of the bijections

$$\text{Hom}_{\mathcal{C}_S}(QX, Y) = \varinjlim_{t:Y \rightarrow Y'} \text{Hom}_{\mathcal{C}}(X, Y') = \text{Hom}_{\text{Ind}\mathcal{C}}(iX, \varinjlim_{t:Y \rightarrow Y'} Y') = \text{Hom}_{\text{Ind}\mathcal{C}}(iX, r_S Y) .$$

The dual assertion is expressed by the bijection

$$\text{Hom}_{\mathcal{C}_S}(Y, QX) = \varprojlim_{s:Y' \rightarrow Y} \text{Hom}_{\mathcal{C}}(Y', X) = \text{Hom}_{\text{Pro}\mathcal{C}}(\varprojlim_{s:Y' \rightarrow Y} Y', iX) = \text{Hom}_{\text{Pro}\mathcal{C}}(l_S Y, iX) .$$

□

1.1.4. DEFINITION. Let $i : \mathcal{C} \rightarrow \text{Ind}\mathcal{C}$ be the canonical fully faithful functor; then we have a natural morphism of functors $\delta_S : i \rightarrow r'_S Q = r'_S$, because for any X in \mathcal{C} we have $\text{id}_X \in S$. The morphism corresponds to the identity of QX under the bijection of ind-adjointness between Q and r_S .

Dually, let $i : \mathcal{C} \rightarrow \text{Pro}\mathcal{C}$ be the canonical fully faithful functor; then we have a natural morphism of functors $\sigma_S : l_S Q \rightarrow i$, corresponding to the identity of QX under the bijection of pro-adjointness between Q and l_S .

For any X object of \mathcal{C} , the morphism $\delta_S(X) : i(X) \rightarrow r'_S(X)$ is represented by the identity of X as a morphism between ind-objects, while as functors it is identified as the canonical morphism

$$\delta_S(X)(Z) : i(X)(Z) = \text{Hom}_{\mathcal{C}}(Z, X) \longrightarrow \varinjlim_{s:X \rightarrow X'} \text{Hom}_{\mathcal{C}}(Z, X') = r'_S(X)(Z)$$

since the left hand side appears in the inductive limit of the right hand side.

1.1.5. DEFINITION. We say that an object X of \mathcal{C} is inert for r_S or right inert for S if $\delta_S(X)$ is an isomorphism; it is right localizable with respect to S if $r'_S(X)$ is representable. Put $R_S(X) := \varinjlim_{X/S} X'$; then X is right localizable with respect to S if and only if the canonical morphism $r'_S(X) \rightarrow R_S(X)$ is an isomorphism in $\text{Ind}\mathcal{C}$.

Dually, we say that an object X of \mathcal{C} is inert for l_S or left inert for S if $\sigma_S(X)$ is an isomorphism; it is left localizable with respect to S if $l'_S(X)$ is representable. Put $L_S(X) := \varprojlim_{S/X} X'$; then X is left localizable with respect to S if and only if the canonical morphism $L_S(X) \rightarrow l'_S(X)$ is an isomorphism in $\text{Pro}\mathcal{C}$.

The criterion 0.1.3 says that X is right localizable with respect to S if and only if there exists $s_0 : X \rightarrow X_0$ in S and a morphism $t_0 : R_S(X) \rightarrow X_0$ such that $\iota_0 t_0 = \text{id}_{R_S(X)}$ and for any $s : X \rightarrow X'$ there exists an object X'' in S/X with morphisms $f' : X' \rightarrow X''$ and $f_0 : X_0 \rightarrow X''$ in S/X such that $f_0 t_0 \iota' = f'$ (ι_0 and ι' indicate the canonical morphisms from X_0 and X' to $R_S(X)$).

Moreover X is right inert for S if and only if X is right localizable with respect to S and the canonical morphism $X \rightarrow R_S(X)$ is an isomorphism; that is, for any $s : X \rightarrow X'$ in X/S , there exists a morphism $t_{X'} : X' \rightarrow X$ such that: for any morphism $f : X' \rightarrow X''$ in X/S we have $t_{X'} f = t_{X''}$; $t_X = \text{id}_X$; for any $s : X \rightarrow X'$ in X/S there exists $s' : X \rightarrow X''$ in X/S and a morphism $f : X' \rightarrow X''$ in X/S such that $s' t_{X'} = f$.

1.1.6. THEOREM (UNIVERSAL PROPERTY OF LOCALIZING FUNCTORS). For any $G : \mathcal{C}_S \rightarrow \text{Ind}\mathcal{C}$ the map β to $(\beta \bullet Q) \circ \delta_S$ induces a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{C})}(r_S, G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C})}(i, GQ) ,$$

that is, the pair (r_S, δ_S) represents the functor $\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C}) \rightarrow \text{Set}$ which sends G to $\text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C})}(i, GQ)$.

Dually, for any $G : \mathcal{C}_S \rightarrow \text{Pro}\mathcal{C}$ the map β to $\sigma_S \circ (\beta \bullet Q)$ induces a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{C})}(G, l_S) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{C})}(GQ, i) .$$

PROOF. The proof is a consequence of the following more general proposition, applied to $F = \text{id}_{\text{Ind}\mathcal{C}}$. We only notice explicitly that the bijection is the composite of the usual one

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{C})}(r_S, G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C})}(r_S Q, GQ)$$

induced by the horizontal composition with id_Q (β to $\beta \bullet Q$) and the map

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C})}(r'_S, H) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C})}(i, H) .$$

induced by the vertical composition with δ_S (γ to $\gamma \circ \delta_S$), which is a bijection if the functor H sends S to isomorphisms of $\text{Ind}\mathcal{C}$. \square

1.1.7. PROPOSITION. For any functor $F : \text{Ind}\mathcal{C} \rightarrow \text{Ind}\mathcal{D}$, define $r_S(F) := Fr_S$. Then for any $G : \mathcal{C}_S \rightarrow \text{Ind}\mathcal{D}$ the map

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{D})}(r_S(F), G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})}(Fi, GQ)$$

induced by β to $(\beta \bullet Q) \circ \delta_S(F)$, where $\delta_S(F) = F \bullet \delta_S$, is a bijection.

Dually, for any functor $F : \text{Pro}\mathcal{C} \rightarrow \text{Pro}\mathcal{D}$, define $l_S(F) := Fl_S$. Then for any $G : \mathcal{C}_S \rightarrow \text{Pro}\mathcal{D}$ the map

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{D})}(G, l_S(F)) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{D})}(GQ, Fi)$$

induced by β to $\sigma_S(F) \circ (\beta \bullet Q)$, where $\sigma_S(F) = F \bullet \sigma_S$, is a bijection.

PROOF. Note that the map is the composite of the usual bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{D})}(r_S(F), G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})}(r_S(F)Q, GQ)$$

induced by the horizontal composition with id_Q (sending β to $\beta \bullet Q$), and the map

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})}(r'_S(F), H) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})}(Fi, H) .$$

induced by the vertical composition with $\delta_S(F)$ (sending γ to $\gamma \circ \delta_S(F)$), which we will prove to be a bijection if the functor H sends S to isomorphisms of $\text{Ind}\mathcal{D}$ (as is the case for $H = GQ$ since any morphism of S becomes an isomorphism after application of Q). In fact consider a morphism $\alpha : Fi \rightarrow H$; by hypothesis for any X object of \mathcal{C} and any $s : X \rightarrow X'$ in S , we have that $H(s) : H(X) \rightarrow H(X')$ is an isomorphism, therefore the $\{H(s)^{-1}\alpha(X') : Fi(X') \rightarrow H(X)\}$ is a compatible system of morphisms:

$$\begin{array}{ccccc} H(X') & \xleftarrow{\alpha(X')} & Fi(X') & \xrightarrow{\delta_S(F)(X')} & r'_S(F)(X') \\ H(s) \uparrow \cong & & \uparrow Fi(s) & & \cong \uparrow r'_S(F)(s) \\ H(X) & \xleftarrow{\alpha(X)} & Fi(X) & \xrightarrow{\delta_S(F)(X)} & r'_S(F)(X) . \end{array}$$

The definition of $r'_S(F)(X) = \varprojlim Fi(X')$ then gives a canonical morphism $r'_S(F)(X) \rightarrow H(X)$ which uniquely factorizes the given system through the $\{\delta_S(F)(X')\}$. \square

1.1.8. COROLLARY (LOCALIZATIONS AS ADJOINT FUNCTORS). Let S be a saturated multiplicative system in a category \mathcal{C} ; for any category \mathcal{D} , consider the functor

$$Q : \mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{D}) \longrightarrow \mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})$$

given by the composition with the canonical $Q : \mathcal{C} \rightarrow \mathcal{C}_S$.

Then we can define a left adjoint

$$r_S : \mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D}) \longrightarrow \mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{D})$$

of Q in the following way. For any $F : \mathcal{C} \rightarrow \text{Ind}\mathcal{D}$ let $\overline{F} : \text{Ind}\mathcal{C} \rightarrow \text{Ind}\mathcal{D}$ be its canonical extension to $\text{Ind}\mathcal{C}$, and define $r_S(F) := \overline{F} \circ r_S$. Then 1.1.7 proves that the canonical morphism

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{D})}(r_S(F), G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{D})}(F, GQ) .$$

is a bijection for any $G : \mathcal{C}_S \rightarrow \text{Ind}\mathcal{D}$.

Dually, the functor

$$l_S : \mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{D}) \longrightarrow \mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{D})$$

is a right adjoint for the canonical functor

$$Q : \mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{D}) \longrightarrow \mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{D})$$

given by the composition with Q ; in particular for any $G : \mathcal{C}_S \rightarrow \text{Pro}\mathcal{D}$ we have the bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{D})}(G, l_S(F)) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{D})}(GQ, F) .$$

\square

1.2. DEFINITION (DELIGNE LOCALIZED FUNCTORS). Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a functor, and S, S' saturated multiplicative systems in $\mathcal{C}, \mathcal{C}'$ respectively. Then the Deligne right localized functor of F with respect to S and S' is the functor $r_{S,S'}(F) := \text{Ind}(Q'F)r_S : \mathcal{C}_S \longrightarrow \text{Ind}\mathcal{C}'_{S'}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r'_S} \text{Ind}\mathcal{C} & \xrightarrow{\text{Ind}(F)} \text{Ind}\mathcal{C}' \\ Q \downarrow & \nearrow r_S & \downarrow \text{Ind}(Q') \\ \mathcal{C}_S & \xrightarrow{r_{S,S'}(F)} & \text{Ind}\mathcal{C}'_{S'} . \end{array}$$

We will also use the functor $r_{S,S'}(F)Q = \text{Ind}(Q'F)r'_S : \mathcal{C} \longrightarrow \text{Ind}\mathcal{C}'_{S'}$, which will be denoted by $r'_{S,S'}(F)$.

Dually, the Deligne left localized functor of F with respect to S and S' is $l_{S,S'}(F) := \text{Pro}(Q'F)l_S : \mathcal{C}_S \longrightarrow \text{Pro}\mathcal{C}'_{S'}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{l'_S} \text{Pro}\mathcal{C} & \xrightarrow{\text{Pro}(F)} \text{Pro}\mathcal{C}' \\ Q \downarrow & \nearrow l_S & \downarrow \text{Pro}(Q') \\ \mathcal{C}_S & \xrightarrow{l_{S,S'}(F)} & \text{Pro}\mathcal{C}'_{S'} . \end{array}$$

We will also use the functor $l_{S,S'}(F)Q = \text{Pro}(Q'F)l'_S : \mathcal{C} \longrightarrow \text{Pro}\mathcal{C}'_{S'}$, which will be denoted by $l'_{S,S'}(F)$.

1.2.1. We can summarize the situation in the following diagram:

$$\begin{array}{ccccc} & & \text{Ind}\mathcal{C} & \xrightarrow{\text{Ind}(F)} & \text{Ind}\mathcal{C}' \\ & r'_S \nearrow i & \downarrow & \nearrow i' & \downarrow \text{Ind}(Q') \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C}' & & \\ Q \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_S & \xrightarrow{i} & \text{Ind}\mathcal{C}_S & \xrightarrow{\quad} & \text{Ind}\mathcal{C}'_{S'} \\ & & \downarrow & \nearrow i' & \\ & & \mathcal{C}'_{S'} & & \end{array}$$

where we have the following commutativities:

$$\begin{aligned} r_S Q &= r'_S \\ r_{S,S'}(F) &:= \text{Ind}(Q'F)r_S \\ r'_{S,S'}(F) &:= r'_{S,S'}(F)Q = \text{Ind}(Q'F)r_S Q = \text{Ind}(Q'F)r_S \\ \text{Ind}(Q'F)i &= \text{Ind}(Q')i'F = i'Q'F . \end{aligned}$$

We also have the extension of $r_{S,S'}(F)$ to $\text{Ind}\mathcal{C}_S$, which will be denoted $\bar{r}_{S,S'}(F)$, and the extension of $r'_{S,S'}(F)$ to $\text{Ind}\mathcal{C}$, which will be denoted $\bar{r}'_{S,S'}(F)$.

Dually, we have the following diagram:

$$\begin{array}{ccccc} & & \text{Pro}\mathcal{C} & \xrightarrow{\text{Pro}(F)} & \text{Pro}\mathcal{C}' \\ & l'_S \nearrow i & \downarrow & \nearrow i' & \downarrow \text{Pro}(Q') \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C}' & & \\ Q \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_S & \xrightarrow{i} & \text{Pro}\mathcal{C}_S & \xrightarrow{\quad} & \text{Pro}\mathcal{C}'_{S'} \\ & & \downarrow & \nearrow i' & \\ & & \mathcal{C}'_{S'} & & \end{array}$$

where we have the following commutativities:

$$\begin{aligned} l_S Q &= l'_S \\ l_{S,S'}(F) &:= \text{Pro}(Q'F)l_S \\ l'_{S,S'}(F) &:= l_{S,S'}(F)Q = \text{Pro}(Q'F)l_S Q = \text{Pro}(Q'F)l_S \\ \text{Pro}(Q'F)i &= \text{Pro}(Q')i'F = i'Q'F . \end{aligned}$$

We also have the extension of $l_{S,S'}(F)$ to $\text{Pro}\mathcal{C}_S$, which will be denoted $\bar{l}_{S,S'}(F)$, and the extension of $l'_{S,S'}(F)$ to $\text{Pro}\mathcal{C}$, which will be denoted $\bar{l}'_{S,S'}(F)$.

1.2.2. The morphism $\delta_S : i \rightarrow r'_S$ induces a morphism

$$\delta_{S,S'}(F) := \text{Ind}(Q'F) \bullet \delta_S : \text{Ind}(Q'F)i \longrightarrow \text{Ind}(Q'F)r'_S$$

of functors $\text{Ind}(Q'F)i = i'Q'F$ and $r'_{S,S'}(F) : \mathcal{C} \rightarrow \text{Ind}\mathcal{C}'_{S'}$. We denote by $\bar{\delta}_{S,S'}(F)$ the extended morphism between $\text{Ind}(Q'F) = \overline{i'Q'F}$ and $\bar{r}'_{S,S'}(F) : \text{Ind}\mathcal{C} \rightarrow \text{Ind}\mathcal{C}'_{S'}$.

Dually, the morphism $\sigma_S : l'_S \rightarrow i$ induces a morphism

$$\sigma_{S,S'}(F) := \text{Pro}(Q'F) \bullet \sigma_S : \text{Pro}(Q'F)l'_S \longrightarrow \text{Pro}(Q'F)i$$

of functors $l'_{S,S'}(F) : \mathcal{C} \rightarrow \text{Pro}\mathcal{C}'_{S'}$ and $\text{Pro}(Q'F)i = i'Q'F$. We denote by $\bar{\sigma}_{S,S'}(F)$ the extended morphism between $\bar{l}'_{S,S'}(F) : \text{Pro}\mathcal{C} \rightarrow \text{Pro}\mathcal{C}'_{S'}$ and $\text{Pro}(Q'F) = \overline{i'Q'F}$.

1.2.3. DEFINITION. We say that an object X of \mathcal{C} is right inert for F (with respect to S and S') if $\delta_{S,S'}(F)(X)$ is an isomorphism. In particular, right inert objects for S are right inert for any functor (with respect to S and any S').

Dually, we say that an object X of \mathcal{C} is left inert for F (with respect to S and S') if $\sigma_{S,S'}(F)(X)$ is an isomorphism. In particular, left inert objects for S are left inert for any functor (with respect to S and any S').

In the next paragraph we will discuss the property of being localizable for F with respect to S and S' .

1.2.4. THEOREM (UNIVERSAL PROPERTY OF DELIGNE LOCALIZED FUNCTORS). For any $G : \mathcal{C}_S \rightarrow \text{Ind}\mathcal{C}'_{S'}$, the map β to $(\beta \bullet Q) \circ \delta_{S,S'}(F)$ induces a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{C}'_{S'})}(r_{S,S'}(F), G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Ind}\mathcal{C}'_{S'})}(i'Q'F, GQ) .$$

In particular for any $G : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, we have a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Ind}\mathcal{C}'_{S'})}(r_{S,S'}(F), i'G) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(Q'F, GQ)$$

induced in the same way.

Dually, for any $G : \mathcal{C}_S \rightarrow \text{Pro}\mathcal{C}'_{S'}$, the map β to $\sigma_{S,S'}(F) \circ (\beta \bullet Q)$ induces a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{C}'_{S'})}(G, l_{S,S'}(F)) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \text{Pro}\mathcal{C}'_{S'})}(GQ, i'Q'F) .$$

In particular for any $G : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, we have a bijection

$$\text{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \text{Pro}\mathcal{C}'_{S'})}(i'G, l_{S,S'}(F)) \longrightarrow \text{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(GQ, Q'F)$$

induced in the same way.

PROOF. In fact, this is a consequence of 1.1.7. □

1.3. COMPOSITION OF DELIGNE LOCALIZED FUNCTORS. The universal property allows us to find canonical morphisms for the composite of Deligne localized functors; let $F' : \mathcal{C}' \rightarrow \mathcal{C}''$ be another functor and S'' a saturated multiplicative system in \mathcal{C}'' . We then have a canonical morphism:

$$\delta_{S,S',S''}(F', F) : r_{S,S''}(F'F) \longrightarrow \bar{r}_{S',S''}(F')r_{S,S'}(F)$$

given explicitly by

$$\delta_{S,S',S''}(F', F) = \text{Ind}(Q''F') \bullet \bar{\delta}_{S',S''} \bullet \bar{r}_S(F) .$$

In fact the morphism

$$\bar{\delta}_{S',S''}(F') \bullet \delta_{S,S'}(F) : i''Q''F'F \longrightarrow \bar{r}_{S',S''}(F')r_{S,S'}(F)Q$$

factorizes through the canonical morphism $\delta_{S,S''}(F'F) : i''Q''F'F \longrightarrow r_{S,S''}(F'F)Q$.

Dually, we have a canonical morphism:

$$\sigma_{S,S',S''}(F', F) : l_{S,S''}(F'F) \longrightarrow \bar{l}_{S',S''}(F')l_{S,S'}(F)$$

given explicitly by

$$\sigma_{S,S',S''}(F', F) = \text{Pro}(Q''F') \bullet \bar{\sigma}_{S',S''} \bullet \bar{l}_S(F) .$$

The obvious compatibilities induced by uniqueness give the following result.

1.3.1. PROPOSITION. *The Deligne right (resp. left) localization is a normalized lax 2-functor between the 2-category of “categories with multiplicative systems” and the 2-category of “Ind-categories” (resp. “Pro-categories”) sending (\mathcal{C}, S) to $\text{Ind}(\mathcal{C}_S)$ (resp. $\text{Pro}(\mathcal{C}_S)$) and $F : \mathcal{C} \rightarrow \mathcal{C}'$ to the canonical extension of $r_{S,S'}(F)$ (resp. $l_{S,S'}(F)$) to the Ind-category (resp. Pro-category); the canonical morphisms $c_{F',F} := \bar{\delta}_{S,S',S''}(F', F)$ (resp. $c_{F',F} := \bar{\sigma}_{S,S',S''}(F', F)$) give the constraints of composition.*

Remark that the 2-categories are (completely) full subcategories of the 2-category of “categories” (in particular we do not impose any condition on the functors, i.e. the 1-morphisms). We have to verify the compatibility (associativity) of the constraints of composition, i.e. the commutativity of the following diagram

$$\begin{array}{ccc} \bar{\tau}(F''F'F) & \xrightarrow{c_{F'',F'}F} & \bar{\tau}(F'')\bar{\tau}(F'F) \\ \downarrow c_{F'',F',F} & & \downarrow \bar{\tau}(F'') \bullet c_{F',F} \\ \bar{\tau}(F''F')\bar{\tau}(F) & \xrightarrow{c_{F'',F'} \bullet \bar{\tau}(F)} & \bar{\tau}(F'')\bar{\tau}(F')\bar{\tau}(F) \end{array}$$

which is clear because the morphisms involved are defined by universal properties, so that they are unique; and the functoriality at the level of 2-morphisms (i.e. morphisms of functors), which is a tedious but elementary verification. \square

1.4. PROPOSITION. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$ adjoint functors, i.e.*

$$\text{Hom}_{\mathcal{C}'}(FX, X') \cong \text{Hom}_{\mathcal{C}}(X, GX')$$

functorially in objects X of \mathcal{C} and objects X' of \mathcal{C}' ; then the Deligne localized functors $l_{S,S'}(F) : \mathcal{C}_S \rightarrow \text{Pro-}\mathcal{C}'_{S'}$ and $r_{S',S}(G) : \mathcal{C}'_{S'} \rightarrow \text{Ind-}\mathcal{C}_S$ are generalized adjoint functors, i.e., we have

$$\text{Hom}_{\text{Pro}(\mathcal{C}'_{S'})}(l_{S,S'}(F)X, X') \cong \text{Hom}_{\text{Ind}(\mathcal{C}_S)}(X, r_{S,S'}(G)X')$$

functorially in objects X of \mathcal{C} and objects X' of \mathcal{C}' .

PROOF. The proof is an easy consequence of commutativity of inductive limits in the category of sets:

$$\begin{aligned} \text{Hom}_{\text{Pro}(\mathcal{C}'_{S'})}(l_{S,S'}(F)X, X') &= \text{Hom}_{\text{Pro}(\mathcal{C}'_{S'})}\left(\varinjlim_{Y \rightarrow X} FY, X'\right) \\ &= \varinjlim_{Y \rightarrow X} \text{Hom}_{\mathcal{C}'_{S'}}(FY, X') \\ &\cong \varinjlim_{Y \rightarrow X} \varinjlim_{X' \rightarrow Y'} \text{Hom}_{\mathcal{C}'}(FY, Y') \\ &\cong \varinjlim_{X' \rightarrow Y'} \varinjlim_{Y \rightarrow X} \text{Hom}_{\mathcal{C}'}(Y, GY') \\ &\cong \varinjlim_{X' \rightarrow Y'} \text{Hom}_{\mathcal{C}_S}(X, GY') \\ &\cong \text{Hom}_{\text{Ind}(\mathcal{C}_S)}\left(X, \varinjlim_{X' \rightarrow Y'} GY'\right) \\ &= \text{Hom}_{\text{Ind}(\mathcal{C}_S)}(X, r_{S,S'}(G)X'). \end{aligned}$$

\square

1.5. EXAMPLE: THE Hom BIFUNCTOR. Let \mathcal{C} be a category and S a saturated multiplicative system in \mathcal{C} . Consider the bifunctor

$$\text{Hom}_{\mathcal{C}} : \mathcal{C}^o \times \mathcal{C} \longrightarrow \text{Set};$$

its right localized functor is $r_S \text{Hom}_{\mathcal{C}} := \text{Ind}(\text{Hom}_{\mathcal{C}}) \circ r_S$ where

$$\text{Ind}(\text{Hom}_{\mathcal{C}}) : \text{Ind}(\mathcal{C}^o \times \mathcal{C}) \cong \text{Pro}(\mathcal{C})^o \times \text{Ind}(\mathcal{C}) \longrightarrow \text{IndSet}$$

sends (X_i, Y_j) to $\varinjlim_{i,j} \text{Hom}_{\mathcal{C}}(X_i, Y_j)$, and

$$r_S : (\mathcal{C}^o \times \mathcal{C})_{S^o \times S} \longrightarrow \text{Ind}(\mathcal{C}^o \times \mathcal{C}) \cong \text{Pro}(\mathcal{C})^o \times \text{Ind}(\mathcal{C})$$

sends (X, Y) to $(S/X, Y/S)$. Therefore we have

$$r_S \text{Hom}_{\mathcal{C}}(X, Y) = \text{Ind}(\text{Hom}_{\mathcal{C}})(S/X, Y/S) = \varinjlim_{S/X, Y/S} \text{Hom}_{\mathcal{C}}(X', Y')$$

and in particular we obtain that

$$\varinjlim r_S \text{Hom}_{\mathcal{C}} = \text{Hom}_{\mathcal{C}_S} .$$

On the other hand, the left localization is defined as $l_S \text{Hom}_{\mathcal{C}} := \text{Pro}(\text{Hom}_{\mathcal{C}}) \circ l_S$ where

$$\text{Pro}(\text{Hom}_{\mathcal{C}}) : \text{Pro}(\mathcal{C}^o \times \mathcal{C}) \cong \text{Ind}(\mathcal{C})^o \times \text{Pro}(\mathcal{C}) \longrightarrow \text{ProSet}$$

sends (X_i, Y_j) to “ $\varprojlim_{i,j}$ ” $\text{Hom}_{\mathcal{C}}(X_i, Y_j)$, and

$$l_S : (\mathcal{C}^o \times \mathcal{C})_{S^o \times S} \longrightarrow \text{Pro}(\mathcal{C}^o \times \mathcal{C}) \cong \text{Ind}(\mathcal{C})^o \times \text{Pro}(\mathcal{C})$$

sends (X, Y) to $(X/S, S/Y)$. Therefore we have

$$l_S \text{Hom}_{\mathcal{C}}(X, Y) = \text{Pro}(\text{Hom}_{\mathcal{C}})(X/S, S/Y) = \varprojlim_{X/S, S/Y} \text{Hom}_{\mathcal{C}}(X', Y')$$

and in particular we obtain that

$$\varinjlim \circ l_S \text{Hom}_{\mathcal{C}}(X, Y) \cong \varinjlim_{X/S, S/Y} \text{Hom}_{\mathcal{C}}(X', Y') \cong \text{Hom}_{\mathcal{C}}(\varinjlim_{X/S} X', \varinjlim_{S/Y} Y') \cong \text{Hom}_{\mathcal{C}}(\varinjlim r_S X, \varinjlim l_S Y) .$$

2. Grothendieck-Verdier localized functors.

Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor, S and S' be right (resp. left for the dual assertions) quasi-saturated multiplicative systems in \mathcal{C} and \mathcal{C}' , respectively.

2.1. DEFINITION. We say that $R_{S,S'}(F)(X)$ exists, or F is right localizable on X with respect to S and S' , or again that X is right localizable for F with respect to S and S' , if $r_{S,S'}(X)$ is an essentially constant object in $\text{Ind}\mathcal{C}'_{S'}$, i.e. if it is isomorphic in $\text{Ind}\mathcal{C}'_{S'}$ to an object of (the image of) $\mathcal{C}'_{S'}$. This means that there exists $R_{S,S'}(F)(X)$ (necessarily isomorphic to $\varinjlim r_{S,S'}(F)(X) = \varinjlim_{X/S} F(X')$ in $\mathcal{C}'_{S'}$), such that $i' R_{S,S'}(F)(X) \cong r_{S,S'}(F)(X)$.

Dually, we say that $L_{S,S'}(F)(X)$ exists, or F is left localizable on X with respect to S and S' , or again that X is left localizable for F with respect to S and S' , if $l_{S,S'}(X)$ is an essentially constant object in $\text{Pro}\mathcal{C}'_{S'}$, i.e., if it is isomorphic in $\text{Pro}\mathcal{C}'_{S'}$ to an object of (the image of) $\mathcal{C}'_{S'}$. This means that there exists $L_{S,S'}(F)(X)$ (necessarily isomorphic to $\varprojlim l_{S,S'}(F)(X) = \varprojlim_{S/X} F(X')$ in $\mathcal{C}'_{S'}$), such that $i' L_{S,S'}(F)(X) \cong l_{S,S'}(F)(X)$.

Notice that if X is right localizable with respect to S (see the definition 1.1.5), then any F is right localizable on X and $r_{S,S'}F(X) \cong F(R_S(X))$. Moreover, if X is right inert for F with respect to S and S' (see the definition 1.2.3), then (F is right localizable on X and) $r_{S,S'}F(X) \cong F(X)$.

Dually, if X is left localizable with respect to S , then any F is left localizable on X and $l_{S,S'}F(X) \cong F(L_S(X))$. Moreover, if X is left inert for F with respect to S and S' , then (F is left localizable on X and) $l_{S,S'}F(X) \cong F(X)$.

2.2. DEFINITION. We say that the Grothendieck-Verdier localized functor $R_{S,S'}F$ exists if for any object X of \mathcal{C} , F is right localizable on X , i.e. if and only if we have a diagram

$$\begin{array}{ccc} \mathcal{C}_S & \xrightarrow{r_{S,S'}F} & \text{Ind}\mathcal{C}'_{S'} \\ R_{S,S'}F \searrow & & \nearrow i' \\ & \mathcal{C}'_{S'} & \end{array}$$

which is commutative up to an isomorphism of functors; alternatively: $r_{S,S'}(F)$ factorizes through i' up to an isomorphism

$$\varrho_{S,S'}(F) : r_{S,S'}(F) \xrightarrow{\cong} i' R_{S,S'}(F) .$$

Dually, we say that the Grothendieck-Verdier localized functor $L_{S,S'}F$ exists if for any object X of \mathcal{C} , F is left localizable on X , i.e. iff we have a diagram

$$\begin{array}{ccc} \mathcal{C}_S & \xrightarrow{l_{S,S'}F} & \text{Pro}\mathcal{C}'_{S'} \\ L_{S,S'}F \searrow & & \nearrow i' \\ & \mathcal{C}'_{S'} & \end{array}$$

which is commutative up to an isomorphism of functors; alternatively: $l_{S,S'}(F)$ factorizes through i' up to an isomorphism

$$\lambda_{S,S'}(F) : l_{S,S'}(F) \xrightarrow{\cong} i' L_{S,S'}(F) .$$

We now prove that the Grothendieck-Verdier localized functors are characterized by the usual universal properties.

2.3. PROPOSITION. *The right Grothendieck-Verdier localized functor $R_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$ is defined by the following universal property: there exists a canonical morphism*

$$\Delta_{S,S'}(F) : Q'F \longrightarrow R_{S,S'}(F)Q$$

such that for any $G : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, the map

$$\mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathcal{C}'_{S'})}(R_{S,S'}(F), G) \longrightarrow \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(Q'F, GQ)$$

sending α to $(\alpha \bullet Q) \circ \Delta_{S,S'}(F)$ is a bijection.

Dually, the left Grothendieck-Verdier localized functor $L_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$ is defined by the following universal property: there exists a canonical morphism

$$\Sigma_{S,S'}(F) : L_{S,S'}(F)Q \longrightarrow Q'F$$

such that for any $G : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, the map

$$\mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathcal{C}'_{S'})}(G, L_{S,S'}(F)) \longrightarrow \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(GQ, Q'F)$$

sending α to $\Sigma_{S,S'}(F) \circ (\alpha \bullet Q)$ is a bijection.

PROOF. Suppose that $R_{S,S'}(F)$ exists; then there exists an isomorphism $\varrho_{S,S'}(F) : r_{S,S'}(F) \xrightarrow{\cong} i' R_{S,S'}(F)$ and the composition

$$i'Q'F \xrightarrow{\delta_{S,S'}(F)} r_{S,S'}(F)Q \xrightarrow{\varrho_{S,S'}(F) \bullet Q} i' R_{S,S'}(F)Q$$

gives a morphism necessarily of the form $i' \bullet \Delta_{S,S'}(F)$ with $\Delta_{S,S'}(F) : Q'F \longrightarrow R_{S,S'}(F)Q$. This morphism induces the following bijection:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(Q'F, GQ) &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathrm{Ind}(\mathcal{C}'_{S'}))}(r_{S,S'}(F), i'G) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathrm{Ind}(\mathcal{C}'_{S'}))}(r_{S,S'}(F)Q, i'GQ) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathrm{Ind}(\mathcal{C}'_{S'}))}(i' R_{S,S'}(F)Q, i'GQ) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(R_{S,S'}(F)Q, GQ) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathcal{C}'_{S'})}(R_{S,S'}(F), G) \end{aligned}$$

for any $G : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, which proves the universal property. Vice-versa, suppose that there exists a functor $R_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$ endowed with a morphism $\Delta_{S,S'}(F) : Q'F \longrightarrow R_{S,S'}(F)Q$ with the stated universal property; then by the universal property of $r_{S,S'}(F)$, using $G = R_{S,S'}(F)$ we find a canonical morphism

$$\varrho_{S,S'}(F) : r_{S,S'}(F) \longrightarrow i' R_{S,S'}(F)$$

corresponding to $\Delta_{S,S'}(F) : Q'F \longrightarrow R_{S,S'}(F)Q$ by $(\varrho_{S,S'}(F) \bullet Q) \circ \delta_{S,S'}(F) = i' \bullet \Delta_{S,S'}(F)$; we have to prove that it is an isomorphism. We see that the composite bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathrm{Ind}(\mathcal{C}'_{S'}))}(r_{S,S'}(F), i'G) &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}, \mathcal{C}'_{S'})}(Q'F, GQ) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathcal{C}'_{S'})}(R_{S,S'}(F), G) \\ &\cong \mathrm{Hom}_{\mathcal{Funct}(\mathcal{C}_S, \mathrm{Ind}(\mathcal{C}'_{S'}))}(i' R_{S,S'}(F), i'G) \end{aligned}$$

is induced by the composition with $\varrho_{S,S'}(F)$ for any G ; this implies that $\varrho_{S,S'}(F)$ is an isomorphism, so that $R_{S,S'}(F)$ is the Grothendieck-Verdier localized functor of F . \square

The universal property of the right (resp. left) Grothendieck-Verdier localized functor gives, as in the case of Deligne functors, a canonical composition morphism

$$\Delta_{S,S',S''}(F', F) : R_{S,S''}(F'F) \longrightarrow R_{S',S''}(F')R_{S,S'}(F)$$

(resp.

$$\Sigma_{S,S',S''}(F', F) : R_{S,S''}(F'F) \longrightarrow R_{S',S''}(F')R_{S,S'}(F) ,$$

if the terms exist, satisfying the usual associative property. Therefore we have the following result.

2.4. PROPOSITION. *The right (resp. left) Grothendieck-Verdier localization gives a partially defined, normalized lax 2-functor from the 2-category of “categories with multiplicative systems” to the 2-category of “categories” sending (\mathcal{C}, S) to \mathcal{C}_S and $F : \mathcal{C} \rightarrow \mathcal{C}'$ to $R_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$ (resp. to $L_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$).*

We are interested in cases in which the canonical morphisms $\delta_{S,S',S''}(F', F)$ and $\Delta_{S,S',S''}(F', F)$ (resp. $\sigma_{S,S',S''}(F', F)$ and $\Sigma_{S,S',S''}(F', F)$) for the compositions are isomorphisms; in that case we say, with a slight abuse of language, that the 2-functors are strict on F and F' .

2.5. PROPOSITION. *Suppose that $R_{S,S'}(F)$ and $R_{S',S''}(F')$ exist (i.e., that there exist the isomorphisms $\varrho_{S,S'}(F)$ and $\varrho_{S',S''}(F')$); then $c_{F',F}$ is an isomorphism if and only if $R_{S,S''}(F'F)$ exists (i.e. there exists the isomorphism $\varrho_{S,S''}(F'F)$) and $\Delta_{S,S',S''}(F', F)$ is an isomorphism. In particular where R and r are defined as 2-functors, one is strict if and only if the other is.*

Dually, suppose that $L_{S,S'}(F)$ and $L_{S',S''}(F')$ exist (i.e., that there exist the isomorphisms $\gamma_{S,S'}(F)$ and $\gamma_{S',S''}(F')$); then $c_{F',F}$ is an isomorphism if and only if $L_{S,S''}(F'F)$ exists (i.e. there exist the isomorphism $\gamma_{S,S''}(F'F)$) and $\Sigma_{S,S',S''}(F', F)$ is an isomorphism. In particular when L and l are defined, one is strict if and only if the other is.

PROOF. This is an easy consequence of the following commutative diagram

$$\begin{array}{ccc} r_{S,S'}(F'F) & \xrightarrow{\varrho_{S,S''}(F'F)} & i'' R_{S,S''}(F'F) \\ \delta_{S,S',S''}(F', F) \downarrow & & \downarrow i'' \bullet \Delta_{S,S',S''}(F', F) \\ \overline{r}_{S',S''}(F') r_{S,S'}(F) & \xrightarrow{\overline{\varrho}_{S',S''}(F') \varrho_{S,S'}(F)} & i'' R_{S',S''}(F') R_{S,S'}(F) \end{array}$$

□

2.6. EXISTENCE CONDITIONS. Let S be a right (resp. left) saturated multiplicative system in \mathcal{C} , and let \mathcal{B} be a full subcategory of \mathcal{C} ; consider the following conditions:

- (i) \mathcal{B} is right sufficient for S , that is for any X in \mathcal{C} there exists $s : X \rightarrow X'$ in S with X' an object of \mathcal{B} (resp. \mathcal{B} is left sufficient for S , that is for any X in \mathcal{C} there exists $s : X' \rightarrow X$ in S with X' an object of \mathcal{B});
- (ii) for any X' in \mathcal{B} , any $s : X' \rightarrow X$ in S is an isomorphism (resp. any $s : X \rightarrow X'$ in S is an isomorphism);
- (iii) any morphism in S between objects of \mathcal{B} is an isomorphism.

We have that (ii) implies (iii), and (iii) implies that any object of \mathcal{B} is right (resp. inert) inert for S .

The condition (i) implies that

$$r_S(X) \cong \varinjlim_{X/S \cap \mathcal{B}} X' \quad (\text{resp. } l_S(X) \cong \varprojlim_{S \cap \mathcal{B}/X} X')$$

where $X/S \cap \mathcal{B}$ (resp. $S \cap \mathcal{B}/X$) is the full subcategory of X/S (resp. S/X) whose objects are of the form $X \rightarrow X'$ (resp. $X' \rightarrow X$) with X' object of \mathcal{B} . Moreover, if the condition (iii) holds, then any object of \mathcal{C} is right (resp. left) localizable with respect to S , and $r_S(X) \cong X'$ (resp. $l_S(X) \cong X'$) for any $X \rightarrow X'$ (resp. $X' \rightarrow X$) in S with $X' \in \mathcal{B}$; finally, the right (resp. left) inert objects for S are exactly the objects of \mathcal{C} which are isomorphic to some object of \mathcal{B} .

2.6.1. In particular if the conditions (i) and (iii) hold, then any functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is right (resp. left) derivable with respect to S and any S' . In fact $R_{S,S'}(F)X \cong F(X')$ (resp. $L_{S,S'}(F)X \cong F(X')$) for any X' as before.

2.6.2. As usual, we may specify the previous conditions (ii) and (iii) for a given functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ and right (resp. left) quasi-saturated multiplicative systems S and S' of \mathcal{C} and \mathcal{C}' , respectively, as follows:

- (ii(F)) for any X' in \mathcal{B} , and any $s : X' \rightarrow X$ in S (resp. any $s : X \rightarrow X'$ in S), the image $F(s)$ is in S' ;
- (iii) any morphism in S between objects of \mathcal{B} has image (by F) in S' . Clearly, (ii) implies (iii), and (iii) implies that any object of \mathcal{B} is right (resp. inert) inert for F with respect to S .

Moreover, if \mathcal{B} is right (resp. left) sufficient for S , then any object of \mathcal{C} is right (resp. left) localizable for F with respect to S and S' , and $R_{S,S'}(F)(X) \cong F(X')$ (resp. $L_{S,S'}(F)(X) \cong F(X')$) for any $X \rightarrow X'$ (resp. $X' \rightarrow X$) in S with $X' \in \mathcal{B}$.

2.7. PROPOSITION. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$ be adjoint functors, i.e.*

$$\text{Hom}_{\mathcal{C}'}(FX, X') \cong \text{Hom}_{\mathcal{C}}(X, GX')$$

functorially in objects X of \mathcal{C} and objects X' of \mathcal{C}' ; then the Grothendieck-Verdier localized functors $L_{S,S'}(F) : \mathcal{C}_S \rightarrow \mathcal{C}'_{S'}$, and $R_{S',S}(G) : \mathcal{C}'_{S'} \rightarrow \mathcal{C}_S$, if they exist, are adjoint functors, i.e. we have

$$\mathrm{Hom}_{\mathcal{C}'_{S'}}(L_{S,S'}(F)X, X') \cong \mathrm{Hom}_{\mathcal{C}_S}(X, R_{S',S}(G)X')$$

functorially in objects X of \mathcal{C} and X' of \mathcal{C}' .

The proof is an easy consequence of the analogous statement 1.4 for the Deligne localized functors. \square

2.8. As a consequence of the universal properties, or of the discussion of Deligne localized functors, we remark that the functors R_S and L_S , if they exist, are respectively the left and right adjoint of the canonical functor $\mathcal{Funct}(\mathcal{C}_S, \mathcal{D}) \rightarrow \mathcal{Funct}(\mathcal{C}, \mathcal{D})$ given by the composition with $Q : \mathcal{C} \rightarrow \mathcal{C}_S$.

3. The case of Triangulated Categories.

3.1. If \mathcal{T} is a triangulated category, with translation functor T and class of distinguished triangles $\mathcal{T}r$, then the categories $\mathrm{Ind}(\mathcal{T})$ and $\mathrm{Pro}(\mathcal{T})$ admit translation functors $\mathrm{Ind}(T)$ and $\mathrm{Pro}(T)$ resp., and also a special subcategory of the category of (its) triangles, give by the essential image of the categories $\mathrm{Ind}(\mathcal{T}r)$ and $\mathrm{Pro}(\mathcal{T}r)$ resp. in the category of triangles of $\mathrm{Ind}(T)$ and $\mathrm{Pro}(T)$ respectively.

But these notions do not give triangulated structures to $\mathrm{Ind}(\mathcal{T})$ and $\mathrm{Pro}(\mathcal{T})$, because two axioms can be violated: the completion of morphisms of triangles, and the octahedral axiom.

3.1.1. LEMMA. Let Δ be a distinguished triangle in \mathcal{T} ; define Δ/S to be the category whose objects are the morphisms of distinguished triangles $\Delta \rightarrow \Delta'$ with morphisms in S . Then Δ/S is a (right) filtering category and we have a canonical isomorphism $r_S \Delta \cong \varinjlim_{\Delta/S} \Delta'$, so that the image by r_S of a distinguished triangle Δ is a triangle of $\mathrm{Ind}(\mathcal{T})$ which is an inductive system of distinguished triangle of \mathcal{T} ; in particular we have a functor

$$r_S : \mathcal{T}r \longrightarrow \mathrm{Ind}(\mathcal{T}r) \subseteq \mathcal{T}r(\mathrm{Ind} \mathcal{T}) .$$

Dually, define S/Δ to be the category whose objects are the morphisms of distinguished triangles $\Delta' \rightarrow \Delta$ with morphisms in S . Then S/Δ is a left filtering category and $l_S \Delta \cong \varprojlim_{S/\Delta} \Delta'$, so that the image by l_S of a distinguished triangle Δ is a triangle of $\mathrm{Pro}(\mathcal{T})$ which is a projective system of distinguished triangles of \mathcal{T} ; in particular we have a functor

$$l_S : \mathcal{T}r \longrightarrow \mathrm{Pro}(\mathcal{T}r) \subseteq \mathcal{T}r(\mathrm{Pro} \mathcal{T}) .$$

PROOF. The non trivial fact is the property of the category Δ/S , which depends on the properties of a saturated multiplicative system, especially the completion of squares and the right/left equalization of a pair of morphisms. Let $s' : \Delta \rightarrow \Delta'$ and $s'' : \Delta \rightarrow \Delta''$ two objects of Δ/S ; if $\Delta = (X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow TX_1)$, $\Delta' = (X'_1 \rightarrow X'_2 \rightarrow X'_3 \rightarrow TX'_1)$ and so on, we can complete the commutative diagram

$$\begin{array}{ccc} & X_2 & \xrightarrow{s''} X''_2 \\ f \nearrow & \downarrow & \nearrow f'' \\ X_1 & \xrightarrow{\quad} X''_1 & \\ s' \downarrow & & \\ & X'_2 & \\ & \nearrow f' & \\ & X'_1 & \end{array} \quad \text{to} \quad \begin{array}{ccc} & X_2 & \xrightarrow{s''} X''_2 \\ f \nearrow & \downarrow & \nearrow f'' \\ X_1 & \xrightarrow{\quad} X''_1 & \downarrow u'' \\ s' \downarrow & & \\ & X'_2 & \downarrow \\ & \nearrow f' & \\ & X'_1 & \xrightarrow{u'} A \end{array}$$

where the morphisms named u are in S ; then we can complete the lower side of the cube to a commutative square

$$\begin{array}{ccc} & X'_2 & \xrightarrow{u'} B \xrightarrow{u} C \\ f' \nearrow & & \nearrow g \\ X'_1 & \xrightarrow{u'} & A \end{array}$$

with again u in S ; eventually after composition with a morphism $v : C \rightarrow D$ in S , we obtain a commutative cube

$$\begin{array}{ccccc}
& & X_2 & \xrightarrow{s''} & X_2'' \\
& f \nearrow & \downarrow & f'' \nearrow & \downarrow u'' \\
X_1 & \xrightarrow{\quad} & X_1'' & & \\
s' \downarrow & & \downarrow & & \\
& X_2' & \xrightarrow{\quad} & D & \\
& \nearrow f' & \downarrow & \nearrow h & \\
X_1' & \xrightarrow[u']{\quad} & A & &
\end{array}$$

where horizontal and vertical morphisms are in S . Completing the distinguished triangle on h as $\Gamma = (A \rightarrow D \rightarrow E \rightarrow TA)$, and the morphisms of the cube to morphisms of triangles, we find a diagram of distinguished triangles

$$\begin{array}{ccc}
\Delta & \xrightarrow{s''} & \Delta'' \\
s' \downarrow & & \downarrow u'' \\
\Delta' & \xrightarrow[u']{\quad} & \Gamma
\end{array}$$

in which however the third square

$$\begin{array}{ccc}
X_3 & \longrightarrow & X_3'' \\
\downarrow & & \downarrow \\
X_3' & \longrightarrow & E
\end{array}$$

need not be commutative; we can choose two morphisms $v', v'' : E \rightarrow X_3'''$ in S such that $v''u''s'' = v'u's'$ and the two compositions $D \rightarrow E \rightarrow X_3'''$ coincide (because in any case they coincide after composition on the right with a morphism of S). Finally we can take $X_2''' = D \rightarrow X_3'''$, complete the distinguished triangle Δ''' on this morphism, and complete the morphisms of distinguished triangles $\Gamma \rightarrow \Delta'''$ to obtain a commutative square

$$\begin{array}{ccc}
\Delta & \xrightarrow{s''} & \Delta'' \\
s' \downarrow & & \downarrow u'' \\
\Delta' & \xrightarrow[u']{\quad} & \Delta'''
\end{array}$$

i.e. an object $\Delta \rightarrow \Delta'''$ of Δ/S with two morphisms $\Delta' \rightarrow \Delta'''$ and $\Delta'' \rightarrow \Delta'''$.

A similar argument, starting with two morphisms $f, g : \Delta' \rightarrow \Delta''$ in Δ/S , shows that there exists $\Delta \rightarrow \Delta'''$ in Δ/S with a morphism $\Delta'' \rightarrow \Delta'''$ of Δ/S which equalizes the two given maps. \square

3.2. PROPOSITION. *Let $\Delta = (X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow TX_1)$ be a distinguished triangle in \mathcal{T} ;*

(o) if X_1 and X_3 are right (resp. left) inert or localizable with respect to S , then so too is X_2 .

Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be a triangulated functor of triangulated categories endowed with null systems $\mathcal{N}, \mathcal{N}'$. Then the functor $r_{S,S'}(F)$ (resp. $l_{S,S'}(F)$) sends $\mathcal{T}r(\mathcal{T}/\mathcal{N})$ in $\text{Ind}(\mathcal{T}r(\mathcal{T}'/\mathcal{N}'))$ (resp. $\text{Pro}(\mathcal{T}r(\mathcal{T}'/\mathcal{N}'))$) and

(i) if X_1 and X_3 are right (resp. left) inert for F with respect to S and S' , then so too is X_2 .

(ii) if $R_{S,S'}(F)$ (resp. $L_{S,S'}(F)$) is defined in X_1 and X_3 , then it is defined also in X_2 and

$$R_{S,S'}(F)\Delta := (R_{S,S'}(F)X_1 \rightarrow R_{S,S'}(F)X_2 \rightarrow R_{S,S'}(F)X_3 \rightarrow R_{S,S'}(F)TX_1)$$

(resp.

$$L_{S,S'}(F)\Delta := (L_{S,S'}(F)X_1 \rightarrow L_{S,S'}(F)X_2 \rightarrow L_{S,S'}(F)X_3 \rightarrow L_{S,S'}(F)TX_1) \quad)$$

is a distinguished triangle of $\mathcal{T}'/\mathcal{N}'$.

PROOF. The arguments are a slight modification of Deligne [SGA4,XVII,1.2.2]. For (i) consider the canonical morphism $F(\Delta) \rightarrow rF(\Delta)$; the first is a distinguished triangle, and the second is an ind-object of distinguished triangles. Moreover the first and third morphisms are iso. Therefore for any W we have a morphism of long exact sequences

$$\text{Hom}_{\mathcal{T}}(W, F(\Delta)) \longrightarrow \text{Hom}_{\mathcal{T}}(W, rF(\Delta)) = \varinjlim_{\Delta/S} \text{Hom}_{\mathcal{T}}(W, F(\Delta'))$$

in which we can apply the five lemma. For (ii) we consider the distinguished triangle constructed from the morphism $RF(X_3) \rightarrow RF(TX_1)$ and the morphisms

$$\begin{array}{ccccc} RF(X_1) & \longrightarrow & W & \longrightarrow & RF(X_3) \\ \cong \downarrow & & & & \downarrow \cong \\ rF(X_1) & \longrightarrow & rF(X_2) & \longrightarrow & rF(X_3) \end{array}$$

where the second line is again an ind-object of distinguished triangles; so we can extend the diagram to a morphism of triangles, which is necessarily an isomorphism. The point (o) can be deduced from (i) and (ii) using $F = \text{id}_{\mathcal{T}}$. \square

3.3. PROPOSITION. *Let $F : \mathcal{T} \rightarrow \mathcal{A}$ be a cohomological functor between a triangulated category \mathcal{T} endowed with a null systems \mathcal{N} , and an abelian category \mathcal{A} . Then the functor $r_S(F) : \mathcal{T}/\mathcal{N} \rightarrow \text{Ind}(\mathcal{A})$ (resp. $l_S(F) : \mathcal{T}/\mathcal{N} \rightarrow \text{Pro}(\mathcal{A})$) is a cohomological functor.*

PROOF. Elementary. \square

4. The case of Derived Categories.

4.1. Any additive functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ of abelian categories extends to an additive functor $F = \mathbf{C}^*(F) : \mathbf{C}^*(\mathcal{A}) \rightarrow \mathbf{C}^*(\mathcal{A}')$ and to a triangulated functor $F = \mathbf{K}^*(F) : \mathbf{K}^*(\mathcal{A}) \rightarrow \mathbf{K}^*(\mathcal{A}')$ of categories of homotopic complexes. Localizing with respect to the systems of quasi-isomorphisms of $\mathbf{C}^*(\mathcal{A})$ and $\mathbf{C}^*(\mathcal{A}')$, we have the right Deligne derived functor

$$r(F) = r(\mathbf{K}^*(F)) : \mathbf{D}^*(\mathcal{A}) \longrightarrow \text{Ind}(\mathbf{D}^*(\mathcal{A}'))$$

defined by the composition

$$\mathbf{D}^*(\mathcal{A}) \xrightarrow{r} \text{Ind}(\mathbf{K}^*(\mathcal{A})) \xrightarrow{\text{Ind}(\mathbf{K}^*(F))} \text{Ind}(\mathbf{K}^*(\mathcal{A}')) \xrightarrow{\text{Ind}(Q')} \text{Ind}(\mathbf{D}^*(\mathcal{A}'))$$

having the universal property

$$\text{Hom}_{\mathcal{Funct}(\mathbf{D}^*(\mathcal{A}), \text{Ind} \mathbf{D}^*(\mathcal{A}'))}(r(F), G) \xrightarrow{\cong} \text{Hom}_{\mathcal{Funct}(\mathbf{K}^*(\mathcal{A}), \text{Ind} \mathbf{D}^*(\mathcal{A}'))}(i'Q'F, GQ),$$

induced by the canonical morphism $\delta(F) : i'Q'F \rightarrow r(F)Q$.

Dually we have the left Deligne derived functor

$$l(F) = l(\mathbf{K}^*(F)) : \mathbf{D}^*(\mathcal{A}) \longrightarrow \text{Pro}(\mathbf{D}^*(\mathcal{A}'))$$

defined by the composition

$$\mathbf{D}^*(\mathcal{A}) \xrightarrow{l} \text{Pro}(\mathbf{K}^*(\mathcal{A})) \xrightarrow{\text{Pro}(\mathbf{K}^*(F))} \text{Pro}(\mathbf{K}^*(\mathcal{A}')) \xrightarrow{\text{Pro}(Q')} \text{Pro}(\mathbf{D}^*(\mathcal{A}'))$$

having the universal property

$$\text{Hom}_{\mathcal{Funct}(\mathbf{D}^*(\mathcal{A}), \text{Pro} \mathbf{D}^*(\mathcal{A}'))}(G, l(F)) \xrightarrow{\cong} \text{Hom}_{\mathcal{Funct}(\mathbf{K}^*(\mathcal{A}), \text{Pro} \mathbf{D}^*(\mathcal{A}'))}(GQ, i'Q'F),$$

induced by the canonical morphism $\sigma(F) : l(F)Q \rightarrow i'Q'F$.

4.1.1. We study the condition for a complex to be right (resp. left) inert for the multiplicative system of quasi-isomorphisms. A sufficient condition is to be homotopically equivalent to a complex whose terms are injective (resp. projective) objects of the abelian category \mathcal{A} . The condition is also necessary if the category \mathcal{A} has enough injective (resp. projective) objects. In fact the subcategory of these objects satisfies the conditions of 2.6 (see for example [RD,I.4.5]).

4.1.2. We have also the notion of right and left Grothendieck-Verdier derived functors which are defined if the corresponding Deligne derived functors take their image in the essentially constant objects.

4.2. As an example we consider the Hom^\bullet functor:

$$\text{Hom}^\bullet : \mathbf{C}^b(\mathcal{A})^o \times \mathbf{C}^b(\mathcal{A}) \longrightarrow \mathbf{C}^b(\mathcal{A})$$

for which we have $H^0(\text{Hom}^\bullet(X, Y)) = \text{Hom}_{\mathbf{K}^b(\mathcal{A})}(X, Y)$. The right Deligne derived functor is defined as

$$r\text{Hom}^\bullet := \text{Ind}(\text{Hom}^\bullet)r : \mathbf{D}^b(\mathcal{A})^o \times \mathbf{D}^b(\mathcal{A}) \longrightarrow \text{Ind}(\mathbf{D}^b(\mathcal{A}))$$

sending (X, Y) to $\varinjlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}^\bullet(X', Y')$. Composition with the inductive limit commutes with passage to cohomology and we have

$$H^0(\varinjlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}^\bullet(X', Y')) \cong \varinjlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}_{\mathbf{K}^b(\mathcal{A})}(X', Y') = \text{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, Y) .$$

By contrast, the left Deligne derived functor is defined as

$$l\text{Hom}^\bullet := \text{Pro}(\text{Hom}^\bullet)l : \mathbf{D}^b(\mathcal{A})^o \times \mathbf{D}^b(\mathcal{A}) \longrightarrow \text{Pro}(\mathbf{D}^b(\mathcal{A}))$$

sending (X, Y) to $\varprojlim_{X/\text{qis}, \text{qis}/Y} \text{Hom}^\bullet(X', Y')$. The zero-cohomology gives

$$H^0(\varprojlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}^\bullet(X', Y')) \cong \varprojlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}_{\mathbf{K}^b(\mathcal{A})}(X', Y') .$$

while composition with the projective limit gives

$$\varprojlim l\text{Hom}^\bullet(X, Y) = \varprojlim_{\text{qis}/X, Y/\text{qis}} \text{Hom}^\bullet(X', Y') = \text{Hom}^\bullet(\varinjlim(\text{qis}/X), \varinjlim(Y/\text{qis})) .$$

4.3. COMPARISON BETWEEN $\mathbf{D}^*(\text{Ind}(\mathcal{A}))$ AND $\text{Ind}(\mathbf{D}^*(\mathcal{A}))$. Since $\text{Ind}(\mathcal{A})$ and $\text{Pro}(\mathcal{A})$ are abelian categories if \mathcal{A} is, we may work out the usual derived categories $\mathbf{D}^*(\text{Ind}(\mathcal{A}))$ and $\mathbf{D}^*(\text{Pro}(\mathcal{A}))$ which are triangulated categories, in contrast with $\text{Ind}(\mathbf{D}^*(\mathcal{A}))$ and $\text{Pro}(\mathbf{D}^*(\mathcal{A}))$ which are not, but are involved in the definition of Deligne derived functors. We want to make explicit the relations between these categories when $*$ = b (bounded complexes). We recall the following result of Deligne (see [RD, App., prop. 3]).

4.3.1. LEMMA. *The family of functors $H^p := \text{Ind}H^p : \text{Ind}(\mathbf{D}^b(\mathcal{A})) \rightarrow \text{Ind}Ab$ is conservative, that is a morphism φ in $\text{Ind}(\mathbf{D}^b(\mathcal{A}))$ is an isomorphism in $\text{Ind}(\mathbf{D}^b(\mathcal{A}))$ if and only if for every p the morphism $H^p(\varphi)$ is an isomorphism in $\text{Ind}Ab$.*

4.3.2. The canonical functor $\mathcal{A} \rightarrow \mathbf{C}^*(\mathcal{A})$ extends to a functor $J : \text{Ind}(\mathcal{A}) \rightarrow \text{Ind}(\mathbf{C}^*(\mathcal{A}))$, and using the representation 0.1.12 for the morphisms in the Ind -categories, for $*$ = b we may extend this to a functor $\mathbf{C}^b(\text{Ind}(\mathcal{A})) \rightarrow \text{Ind}(\mathbf{C}^b(\mathcal{A}))$. Note that an object in $\mathbf{C}^b(\text{Ind}(\mathcal{A}))$ must be represented with “commutation relations” $d^2 = 0$ (it is a complex), where the zero morphisms of ind -objects are represented with all components the zero morphisms of the category \mathcal{A} ; this guarantees that the result of the parallelization process is an inductive system of complexes.

Using the Deligne lemma, we may define a functor $J : \mathbf{D}^b(\text{Ind}(\mathcal{A})) \rightarrow \text{Ind}(\mathbf{D}^b(\mathcal{A}))$ making the following diagram commutative

$$\begin{array}{ccc} \mathbf{C}^b(\text{Ind}(\mathcal{A})) & \xrightarrow{J} & \text{Ind}(\mathbf{C}^b(\mathcal{A})) \\ Q \downarrow & & \downarrow \text{Ind}(Q) \\ \mathbf{D}^b(\text{Ind}(\mathcal{A})) & \xrightarrow{J} & \text{Ind}(\mathbf{D}^b(\mathcal{A})) \end{array}$$

We remark that the essential image of $\mathbf{D}^b(\text{Ind}(\mathcal{A}))$ in $\text{Ind}(\mathbf{D}^b(\mathcal{A}))$ consists of Ind -objects in the category of complexes whose cohomology is *uniformly* bounded, that is, not only is it an inductive system of bounded cohomology complexes, but the bound is uniform on the whole of the system. This remark motivates the following definitions.

4.3.3. DEFINITION. *Let $\text{ind}(\mathbf{C}^b(\mathcal{A}))$ (resp. $\text{ind}(\mathbf{K}^b(\mathcal{A}))$, $\text{ind}(\mathbf{D}^b(\mathcal{A}))$) be the full subcategory of $\text{Ind}(\mathbf{C}^b(\mathcal{A}))$ (resp. $\text{Ind}(\mathbf{K}^b(\mathcal{A}))$, $\text{Ind}(\mathbf{D}^b(\mathcal{A}))$) whose objects are inductive systems of complexes X_α such that there exists $N \in \mathbb{N}$ with $X_{\alpha,i} = 0$ (resp. $X_{\alpha,i} = 0$, $H^i(X_\alpha) = 0$) for all $i \notin [-N, N]$ and all α .*

Dually, $\text{pro}(\mathbf{C}^b(\mathcal{A}))$, $\text{pro}(\mathbf{K}^b(\mathcal{A}))$, $\text{pro}(\mathbf{D}^b(\mathcal{A}))$ are the full subcategories of $\text{Pro}(\mathbf{C}^b(\mathcal{A}))$, $\text{Pro}(\mathbf{K}^b(\mathcal{A}))$, $\text{Pro}(\mathbf{D}^b(\mathcal{A}))$ having uniformly limited complexes, or cohomology.

So we have the following canonical functors:

$$\begin{array}{ccc} \mathbf{C}^b(\text{Ind}(\mathcal{A})) & \xrightarrow{J} & \text{ind}(\mathbf{C}^b(\mathcal{A})) & \mathbf{C}^b(\text{Pro}(\mathcal{A})) & \xrightarrow{J} & \text{pro}(\mathbf{C}^b(\mathcal{A})) \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \mathbf{K}^b(\text{Ind}(\mathcal{A})) & \longrightarrow & \text{ind}(\mathbf{K}^b(\mathcal{A})) & \mathbf{K}^b(\text{Pro}(\mathcal{A})) & \longrightarrow & \text{pro}(\mathbf{K}^b(\mathcal{A})) \\ Q \downarrow & & \downarrow \text{Ind}(Q) & Q \downarrow & & \downarrow \text{Ind}(Q) \\ \mathbf{D}^b(\text{Ind}(\mathcal{A})) & \xrightarrow{J} & \text{ind}(\mathbf{D}^b(\mathcal{A})) & \mathbf{D}^b(\text{Pro}(\mathcal{A})) & \xrightarrow{J} & \text{pro}(\mathbf{D}^b(\mathcal{A})) . \end{array}$$

Remark however that the second and third horizontal functors do not have, in general, good properties (see [KS3]).

4.3.4. Similarly, the canonical functor $\mathcal{A} \rightarrow \text{Ind}\mathcal{A}$ extends to the complexes $\mathbf{C}^*(\mathcal{A}) \rightarrow \mathbf{C}^*(\text{Ind}\mathcal{A})$, and trivially we have an extension $\text{ind}(\mathbf{C}^b(\mathcal{A})) \rightarrow \mathbf{C}^*(\text{Ind}\mathcal{A})$ since any inductive system of uniformly bounded complexes can be rewritten as a bounded complex of inductive systems. In fact the functor J is the Ind-adjoint (resp. the Pro-adjoint) of the canonical functor $\mathbf{C}^*(\mathcal{A}) \rightarrow \mathbf{C}^*(\text{Ind}\mathcal{A})$ (resp. $\mathbf{C}^*(\mathcal{A}) \rightarrow \mathbf{C}^*(\text{Pro}\mathcal{A})$).

4.3.5. PROPOSITION. *The categories $\text{ind}(\mathbf{C}^b(\mathcal{A}))$ and $\mathbf{C}^b(\text{Ind}(\mathcal{A}))$ are equivalent, and dually $\text{pro}(\mathbf{C}^b(\mathcal{A}))$ is equivalent to $\mathbf{C}^b(\text{Pro}(\mathcal{A}))$*

In fact the two functors previous defined are equivalence quasi-inverses of each other. \square

4.3.6. PROPOSITION. *The natural functor $\mathbf{D}^b(\text{Ind}(\mathcal{A})) \rightarrow \text{ind}(\mathbf{D}^b(\mathcal{A}))$ (resp. $\mathbf{D}^b(\text{Pro}(\mathcal{A})) \rightarrow \text{pro}(\mathbf{D}^b(\mathcal{A}))$) is conservative.*

This is a consequence of 4.3.1. \square

4.3.7. LEMMA. *Consider the inclusions of $\mathbf{K}^b(\mathcal{A})$ in $\text{ind}(\mathbf{K}^b(\mathcal{A}))$ and $\mathbf{K}^b(\text{Ind}\mathcal{A})$ (resp. in $\text{pro}(\mathbf{K}^b(\mathcal{A}))$ and $\mathbf{K}^b(\text{Pro}\mathcal{A})$) and of $\mathbf{D}^b(\mathcal{A})$ in $\text{ind}(\mathbf{D}^b(\mathcal{A}))$ and $\mathbf{D}^b(\text{Ind}\mathcal{A})$ (resp. in $\text{pro}(\mathbf{D}^b(\mathcal{A}))$ and $\mathbf{D}^b(\text{Pro}\mathcal{A})$). Then for any X object of $\mathbf{C}^b(\mathcal{A})$ and any Z in $\mathbf{K}^b(\text{Ind}\mathcal{A})$ (resp. $\mathbf{K}^b(\text{Pro}\mathcal{A})$) we have*

$$\text{Hom}_{\mathbf{K}^b(\text{Ind}\mathcal{A})}(X, Z) \xrightarrow{\cong} \text{Hom}_{\text{ind}(\mathbf{K}^b(\mathcal{A}))}(X, JZ) \quad (\text{resp. } \text{Hom}_{\mathbf{K}^b(\text{Pro}\mathcal{A})}(Z, X) \xrightarrow{\cong} \text{Hom}_{\text{pro}(\mathbf{K}^b(\mathcal{A}))}(JZ, X))$$

and

$$\text{Hom}_{\mathbf{D}^b(\text{Ind}\mathcal{A})}(X, Z) \xrightarrow{\cong} \text{Hom}_{\text{ind}(\mathbf{D}^b(\mathcal{A}))}(X, JZ) \quad (\text{resp. } \text{Hom}_{\mathbf{D}^b(\text{Pro}\mathcal{A})}(Z, X) \xrightarrow{\cong} \text{Hom}_{\text{pro}(\mathbf{D}^b(\mathcal{A}))}(JZ, X)).$$

In particular, the Ind-object JZ as a functor is defined by $\text{Hom}_{\mathbf{K}^b(\text{Ind}\mathcal{A})}(-, Z)$ in $\text{ind}(\mathbf{K}^b(\mathcal{A}))$ and $\text{Hom}_{\mathbf{D}^b(\text{Ind}\mathcal{A})}(-, Z)$ in $\text{ind}(\mathbf{D}^b(\mathcal{A}))$ (resp. the Pro-object JZ as a functor is defined by $\text{Hom}_{\mathbf{K}^b(\text{Pro}\mathcal{A})}(Z, -)$ in $\text{pro}(\mathbf{K}^b(\mathcal{A}))$ and $\text{Hom}_{\mathbf{D}^b(\text{Pro}\mathcal{A})}(Z, -)$ in $\text{pro}(\mathbf{D}^b(\mathcal{A}))$).

In fact we may suppose that Z in $\mathbf{K}^b(\text{Ind}\mathcal{A})$ is already parallelized, and in that case the boundedness condition gives an isomorphism

$$\text{Hom}_{\mathbf{K}^b(\text{Ind}\mathcal{A})}(X, (Z_i)) \cong \varinjlim_i \text{Hom}_{\mathbf{K}^b(\mathcal{A})}(X, Z_i)$$

which is by definition $\text{Hom}_{\text{ind}(\mathbf{K}^b(\mathcal{A}))}(X, JZ)$. \square

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